5

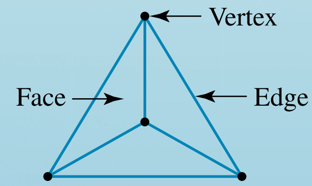
Graphs and networks

5.1 Kick off with CAS

Euler's formula

Leonhard Euler was a Swiss mathematician and physicist who is credited as the founder of graph theory. In graph theory, a graph is made up of vertices (nodes) and edges connecting the vertices.

Euler's formula is considered to be the first theorem of graph theory. It relates to planar graphs — graphs in which there are no intersecting edges. In all planar graphs, edges and vertices divide the graph into a number of faces, as shown in the diagram.



Euler's formula states that in a connected planar graph, $v - e + f = 2$, where v is the number of vertices, e is the number of edges and f is the number of faces in the graph.

1 Using CAS, define and save Euler's formula.

2 Use CAS to solve your formula for the pronumeral f.

3 Use your formula from question 2 to calculate how many faces a planar graph has if it consists of:

  a 5 vertices and 7 edges

  b 4 vertices and 9 edges

  c 3 vertices and 3 edges.

4 Use CAS to solve Euler's formula for the pronumeral e.

5 Use your formula from question 4 to calculate how many edges a planar graph has if it consists of:

  a 6 vertices and 4 faces

  b 8 vertices and 8 faces

  c 7 vertices and 3 faces.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

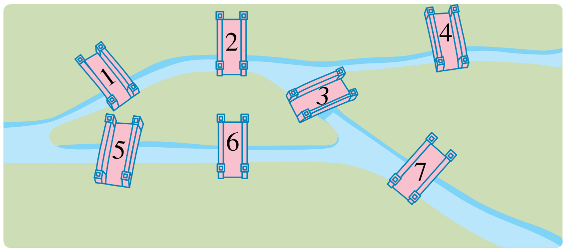
5.2 Definitions and terms

As you will have noticed in previous years, it is a common practice to draw diagrams and other visual and graphic representations when solving many mathematical problems. In the branch of mathematics known as graph theory, diagrams involving points and lines are used as a planning and analysis tool for systems and connections. Applications of graph theory include business efficiency, transportation systems, design projects, building and construction, food chains and communications networks.

Leonhard Euler

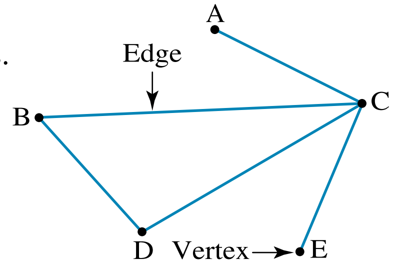
The mathematician Leonhard Euler (1707–83) is usually credited with being the founder of graph theory. He famously used it to solve a problem known as the 'Bridges of Königsberg'. For a long time it had been pondered whether it was possible to travel around the European city of Königsberg (now called Kaliningrad) in such a way that the seven bridges would only have to be crossed once each.

Bridges of Königsberg



Graphs

A **graph** is a series of points and lines that can be used to represent the connections that exist in various settings.



In a graph, the lines are called edges (sometimes referred to as 'arcs') and the points are called vertices (or 'nodes'), with each edge joining a pair of vertices.

Although edges are often drawn as straight lines, they don't have to be.

When vertices are joined by an edge, they are known as 'adjacent' vertices. Note that the edges of a graph can intersect without there being a vertex. For example, the graph at right has five edges and five vertices.

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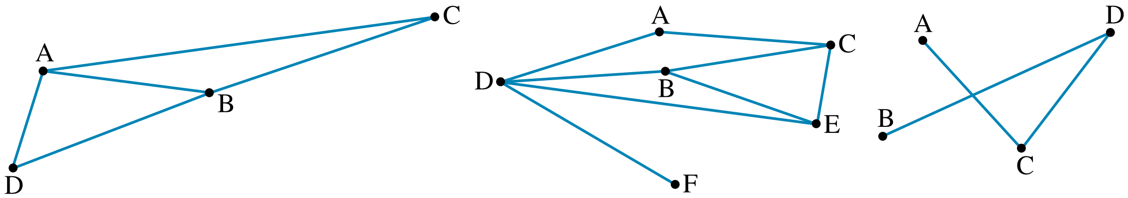
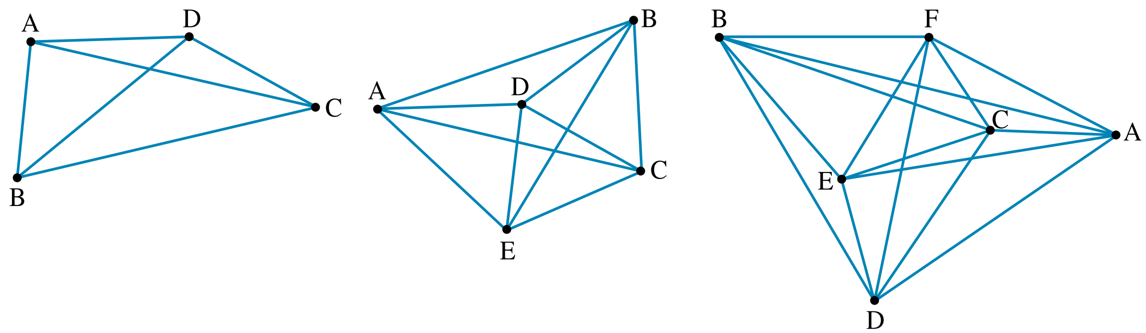
A simple graph is one in which pairs of vertices are connected by one edge at most.

If there is an edge connecting each vertex to all other vertices in the graph, it is called a complete graph.

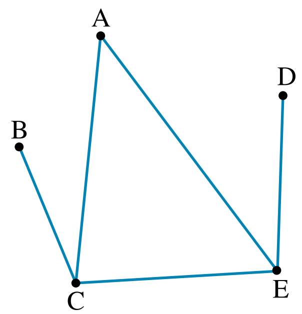
If it is possible to reach every vertex of a graph by moving along the edges, it is called a connected graph; otherwise, it is a disconnected graph.



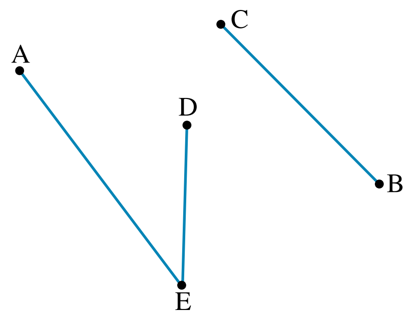
Simple graphs



Complete graphs

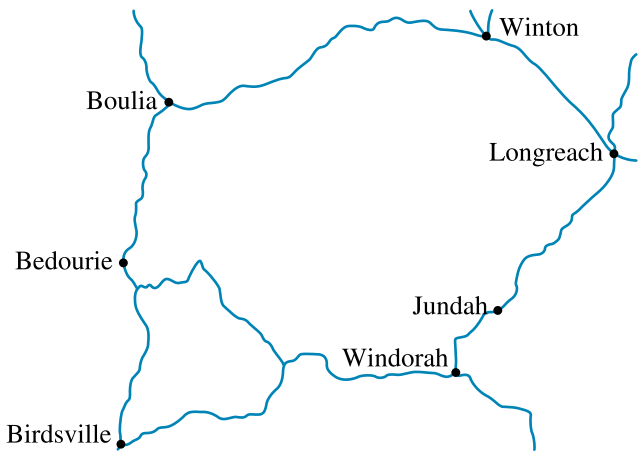


Connected graph

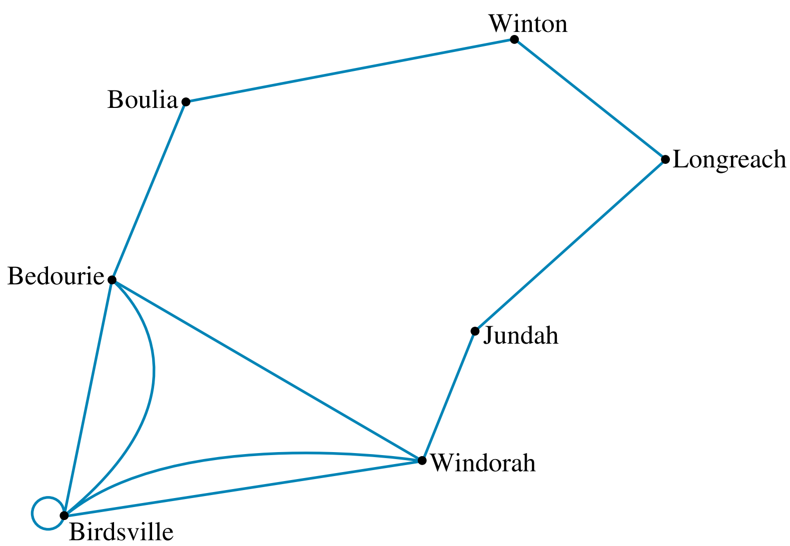


Disconnected graph

Consider the road map shown.



This map can be represented by the following graph.

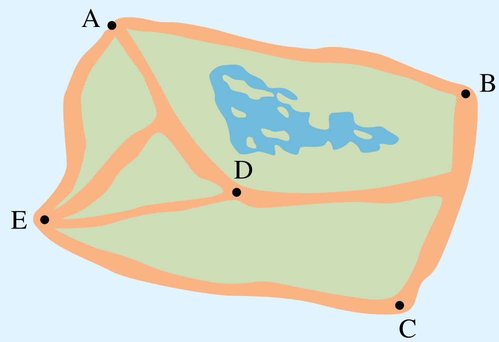


As there is more than one route connecting Birdsville to Windorah and Birdsville to Bedourie, they are each represented by an edge in the graph. In this case we say there are multiple edges. Also, as it is possible to travel along a road from Birdsville that returns without passing through another town, this is represented by an edge. When this happens, the edge is called a **loop**.

If it is only possible to move along the edges of a graph in one direction, the graph is called a directed graph and the edges are represented by arrows. Otherwise it is an undirected graph.

WORKED EXAMPLE 1

The diagram represents a system of paths and gates in a large park. Draw a graph to represent the possible ways of travelling to each gate in the park.



THINK

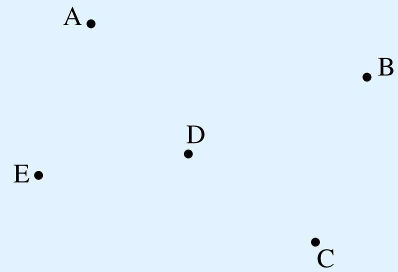
1 Identify, draw and label all possible vertices

2 Draw edges to represent all the direct connections between the identified vertices.

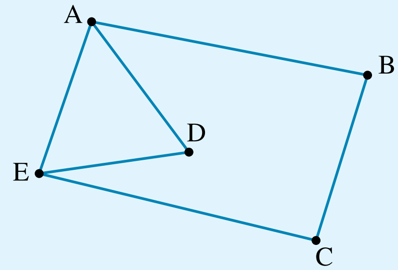
3 Identify all the other unique ways of connecting vertices.

WRITE/DRAW

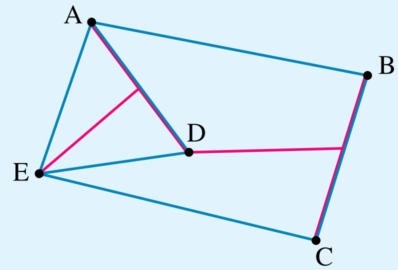
Represent each of the gates as vertices.



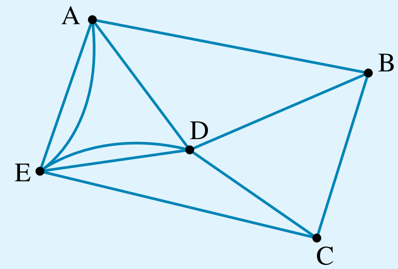
Direct pathways exist for A–B, A–D, A–E, B–C, C–E and D–E.



Other unique pathways exist for A–E, D–E, B–D and C–D.

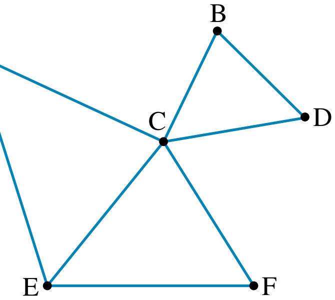


4 Draw the final graph.



The degree of a vertex

When analysing the situation that a graph is representing, it can often be useful to consider the number of edges that are directly connected to a particular vertex. This is referred to as the **degree** of the vertex and is given the notation deg(V), where V represents the vertex.



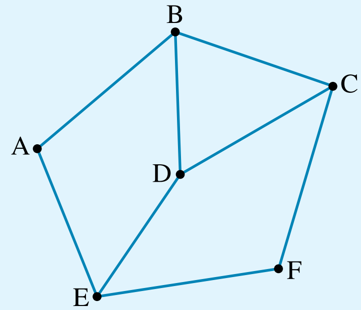
The degree of a vertex = the number of edges directly connected to that vertex.

In the diagram, deg(A) = 2, deg(B) = 2, deg(C) = 5, deg(D) = 2, deg(E) = 3 and deg(F) = 2.

Notice that the sum of the degrees in this graph is 16. The total number of edges in the graph should always be half of the sum of the degrees. In an undirected graph, a vertex with a loop counts as having a degree of 2.

WORKED EXAMPLE 2

For the graph in the following diagram, show that the number of edges is equal to the half sum of the degree of the vertices.



THINK	WRITE
1 Identify the degree of each vertex.	deg(A) = 2, deg(B) = 3, deg(C) = 3, deg(D) = 3, deg(E) = 3 and deg(F) = 2
2 Calculate the sum of the degrees for the graph.	The sum of the degrees for the graph = 2 + 3 + 3 + 3 + 3 + 2 = 16
3 Count the number of edges for the graph.	The graph has the following edges: A–B, A–E, B–C, B–D, C–D, C–F, D–E, E–F. The graph has 8 edges.
4 State the final answer.	The total number of edges in the graph is therefore half the sum of the degrees.

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Units 1 & 2

AOS 3

Topic 2

Concept 2

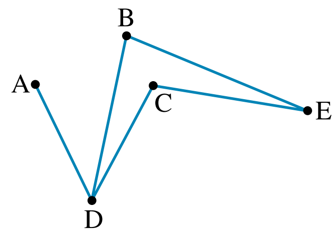
Isomorphic graphs and matrices

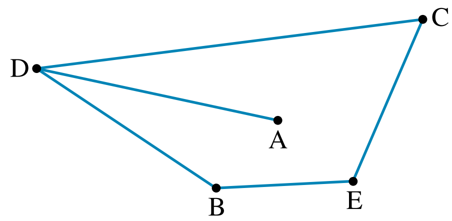
Concept summary

Practice questions

Isomorphic graphs

Consider the following graphs.





For the two graphs, the connections for each vertex can be summarised as shown in the table.

Although the graphs don't look exactly the same, they could be representing exactly the same information. Such graphs are known as **isomorphic graphs**.

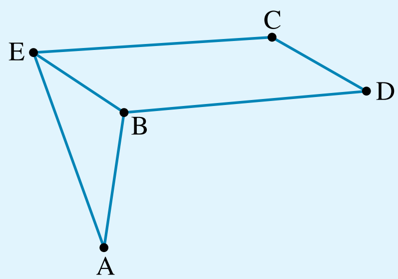
Isomorphic graphs have the same number of vertices and edges, with corresponding vertices having identical degrees and connections.

Vertex	Connections		
A	D		
B	D	E	
C	D	E	
D	A	B	C
E	B	C	

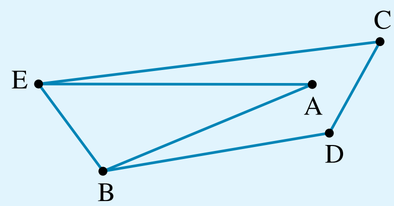
WORKED EXAMPLE 3

Confirm whether the following two graphs are isomorphic.

Graph 1



Graph 2



THINK

1 Identify the degree of the vertices for each graph.

2 Identify the number of edges for each graph.

3 Identify the vertex connections for each graph.

WRITE

Graph	A	B	C	D	E
Graph 1	2	3	2	2	3
Graph 2	2	3	2	2	3

Graph	Edges
Graph 1	6
Graph 2	6

Vertex	Connections		
A	B	E	
B	A	D	E
C	D	E	
D	B	C	
E	A	B	C

4 State the answer.	The two graphs are isomorphic as they have the same number of vertices and edges, with corresponding vertices having identical degrees and connections.

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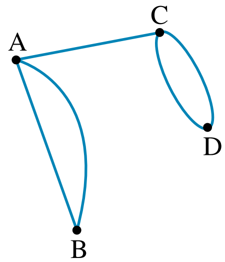
Interactivity
The adjacency matrix
int-6466

Adjacency matrices

Matrices are often used when working with graphs. A matrix that represents the number of edges that connect the vertices of a graph is known as an adjacency matrix.

Each column and row of an adjacency matrix corresponds to a vertex of the graph, and the numbers indicate how many edges are connecting them.



$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Graph — Adjacency matrix

In the adjacency matrix, column 3 corresponds to vertex C and row 4 to vertex D. The '2' indicates the number of edges joining these two vertices.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

Characteristics of adjacency matrices

Adjacency matrices are square matrices with n rows and columns, where 'n' is equal to the number of vertices in the graph.

$$\begin{array}{c} \text{Column: } 1 \quad 2 \quad \dots \quad n-1 \quad n \\ \begin{bmatrix} 0 & 2 & \cdots & 1 & 2 \\ 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 2 \\ 0 & 0 & \cdots & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Row} \\ 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array}$$

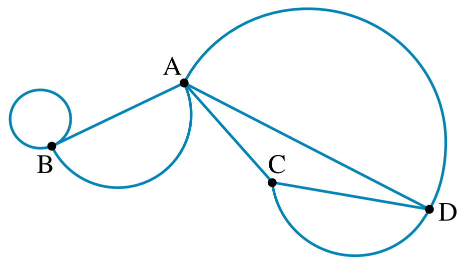
Adjacency matrices are symmetrical around the leading diagonal.

$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Any non-zero value in the leading diagonal will indicate the existence of a loop.

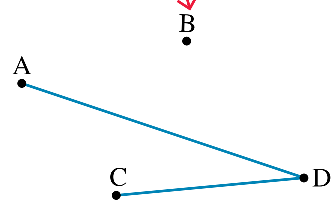
$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & \textcircled{1} & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

The ‘1’ indicates that a loop exists at vertex B:

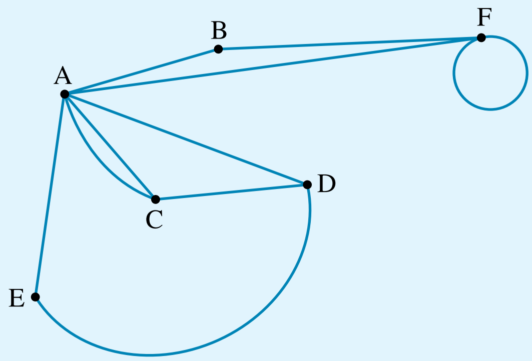


A row consisting of all zeros indicates an isolated vertex (a vertex that is not connected to any other vertex).

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ \boxed{0 \;\; 0 \;\; 0 \;\; 0} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



WORKED EXAMPLE 4 Construct the adjacency matrix for the given graph.



THINK

1 Draw up a table with rows and columns for each vertex of the graph.

WRITE

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

2 Count the number of edges that connect vertex A to the other vertices and record these values in the corresponding space for the first row of the table.

	A	B	C	D	E	F
A	0	1	2	1	1	1

3 Repeat step 2 for all the other vertices.

	A	B	C	D	E	F
A	0	1	2	1	1	1
B	1	0	0	0	0	1
C	2	0	0	1	0	0
D	1	0	1	0	1	0
E	1	0	0	1	0	0
F	1	1	0	0	0	1

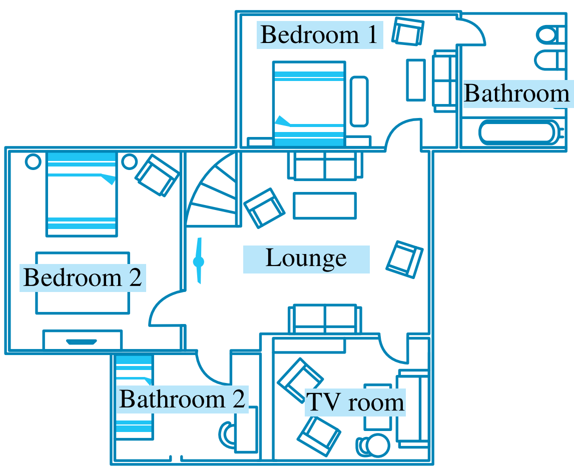
4 Display the numbers as a matrix.

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

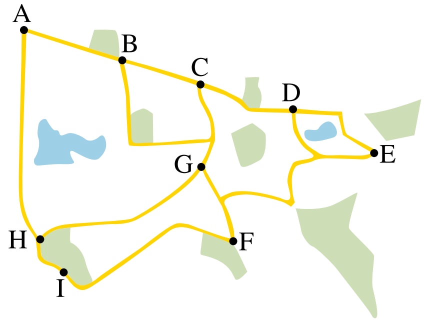
EXERCISE 5.2 Definitions and terms

PRACTISE

1 WE1 The diagram shows the plan of a floor of a house. Draw a graph to represent the possible ways of travelling between each room of the floor.

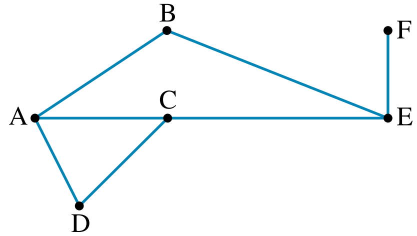


2 Draw a graph to represent the following tourist map.

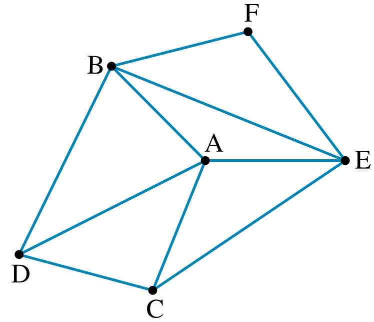


3 WE2 For each of the following graphs, verify that the number of edges is equal to half the sum of the degree of the vertices.

a

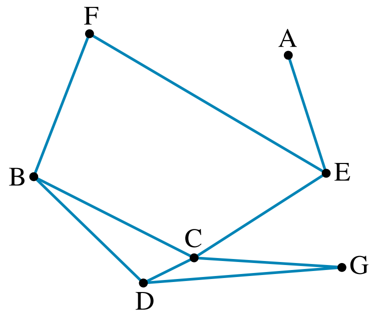


b

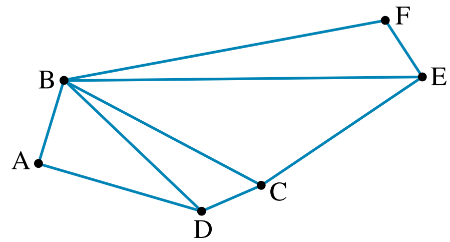


4 For each of the following graphs, verify that the number of edges is equal to half the sum of the degree of the vertices.

a

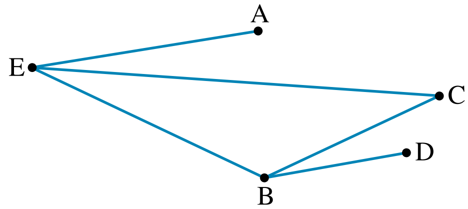


b

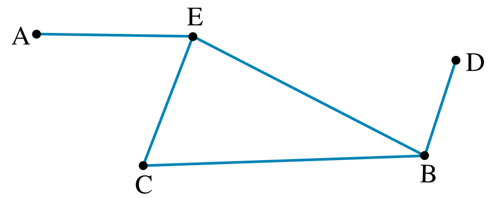


5 WE3 Confirm whether the following pairs of graphs are isomorphic.

a

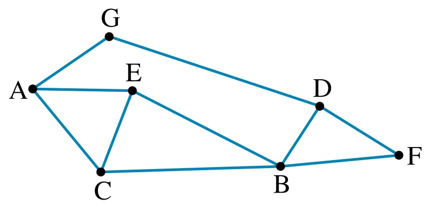


Graph 1

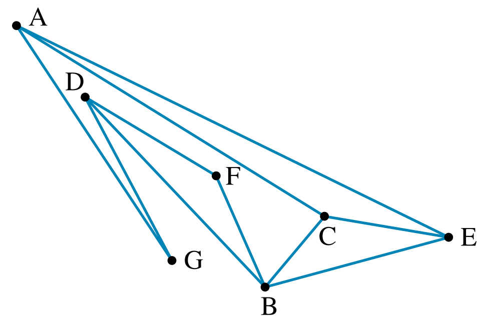


Graph 2

b

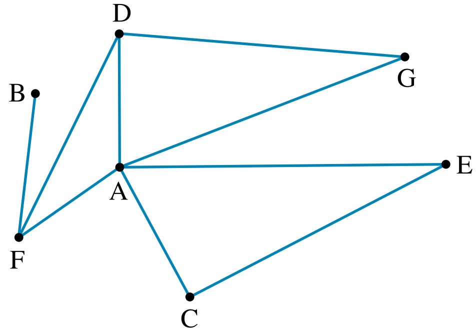


Graph 1

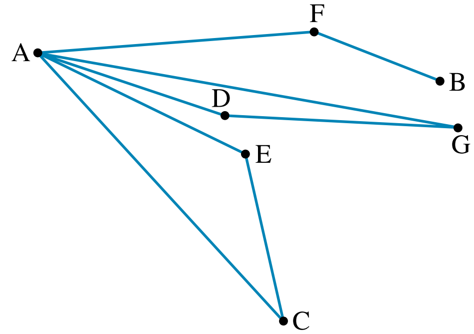


Graph 2

c

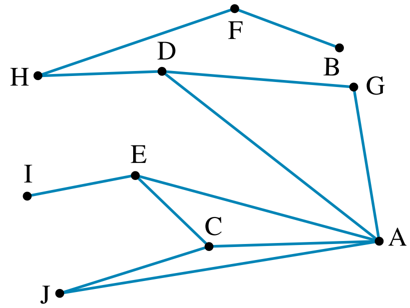


Graph 1

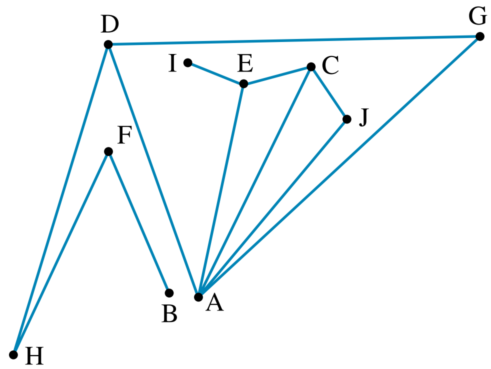


Graph 2

d



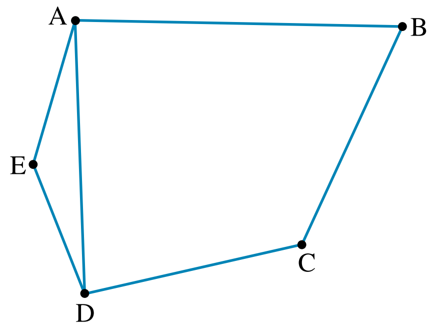
Graph 1



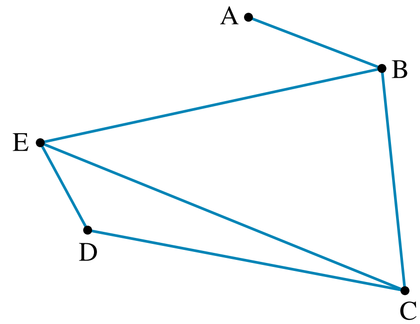
Graph 2

6 Explain why the following pairs of graphs are not isomorphic:

a

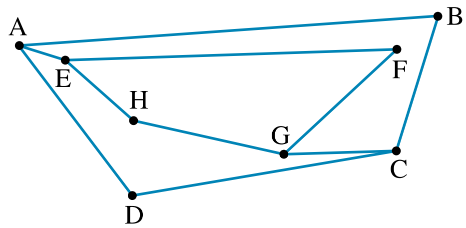


Graph 1

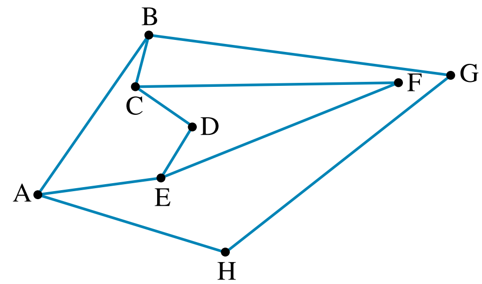


Graph 2

b



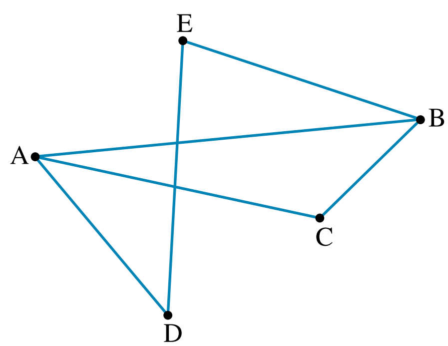
Graph 1



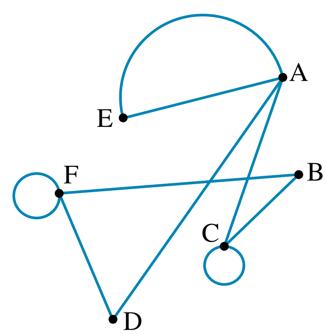
Graph 2

7 **WE4** Construct adjacency matrices for the following graphs.

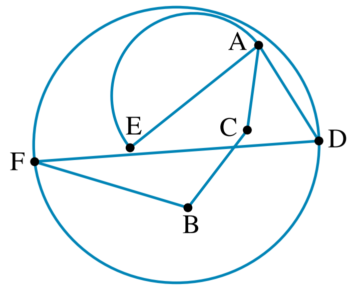
a



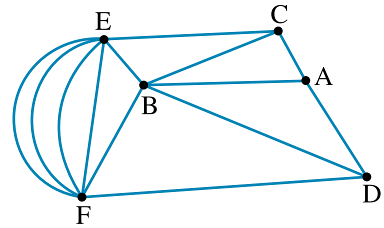
b



c



d



8 Draw graphs to represent the following adjacency matrices.

a $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

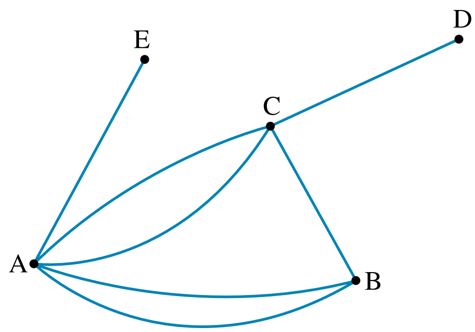
c $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

d $\begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$

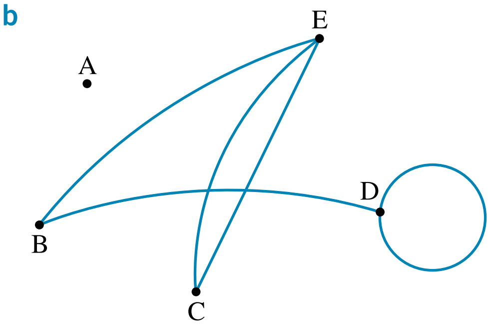
CONSOLIDATE

9 Identify the degree of each vertex in the following graphs.

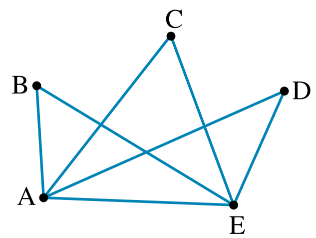
a



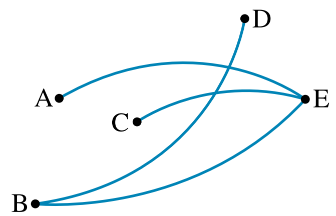
b



c

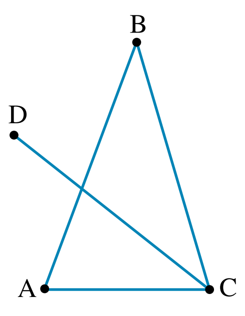


d

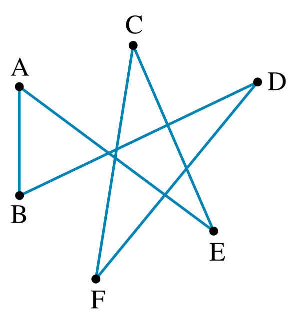


10 Complete the following table for the graphs shown.

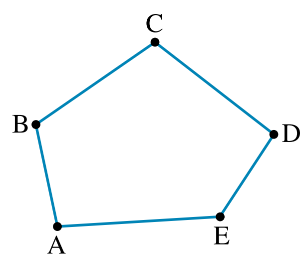
Graph 1



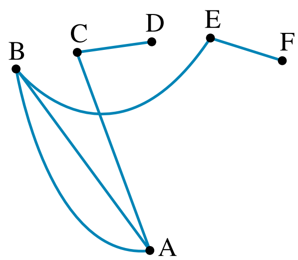
Graph 2



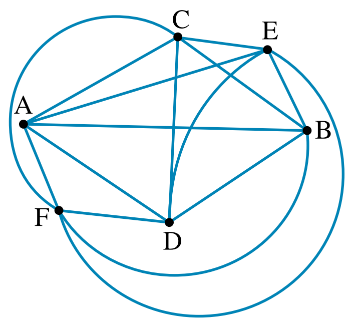
Graph 3



Graph 4



Graph 5

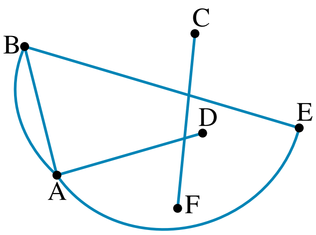


	Simple	Complete	Connected
Graph 1	No	No	Yes
Graph 2			
Graph 3			
Graph 4			
Graph 5			

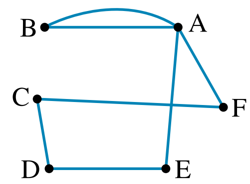
11 Construct the adjacency matrices for each of the graphs shown in question **10**.

12 Identify pairs of isomorphic graphs from the following.

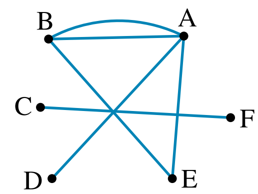
Graph 1



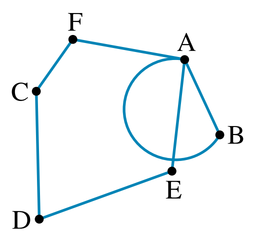
Graph 2



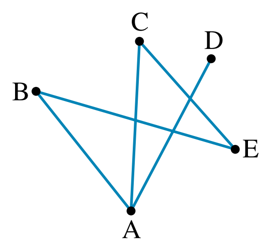
Graph 3



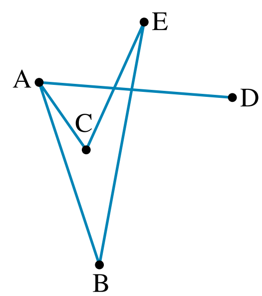
Graph 4



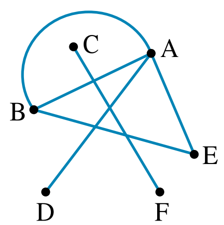
Graph 5



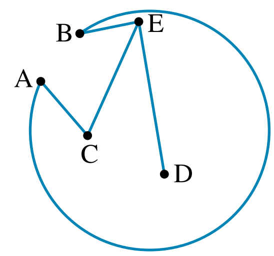
Graph 6



Graph 7



Graph 8



13 Enter details for complete graphs in the following table.

Vertices	Edges
2	
3	
4	
5	
6	
n	

14 Complete the following adjacency matrices.

a $\begin{bmatrix} 0 & 0 & \\ 0 & 2 & 2 \\ 1 & & 0 \end{bmatrix}$

b $\begin{bmatrix} 2 & 1 & & 0 \\ & 0 & & \\ 0 & 1 & 0 & 1 \\ & 2 & & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & & 1 & & 0 \\ 0 & 0 & & & 0 \\ & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & & & 1 & 0 \end{bmatrix}$

d $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ & & & 0 & 0 \\ 0 & & & 0 & 1 \end{bmatrix}$

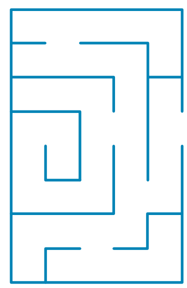
15 Draw a graph of:

a a simple, connected graph with 6 vertices and 7 edges

b a simple, connected graph with 7 vertices and 7 edges, where one vertex has degree 3 and five vertices have degree 2

c a simple, connected graph with 9 vertices and 8 edges, where one vertex has degree 8.

16 By indicating the passages with edges and the intersections and passage endings with vertices, draw a graph to represent the maze shown in the diagram.



Maze

17 Five teams play a round robin competition.

a Draw a graph to represent the games played.

b What type of graph is this?

c What does the total number of edges in the graph indicate?

18 The diagram shows the map of some of the main suburbs of Beijing.

a Draw a graph to represent the shared boundaries between the suburbs.

b Which suburb has the highest degree?

c What type of graph is this?

MASTER

19 The map shows some of the main highways connecting some of the states on the west coast of the USA.

a Draw a graph to represent the highways connecting the states shown.
b Use your graph to construct an adjacency matrix.
c Which state has the highest degree?
d Which state has the lowest degree?

20 Jetways Airlines operates flights in South East Asia.

The table indicates the number of direct flights per day between key cities.

From: / To:	Bangkok	Manila	Singapore	Kuala Lumpur	Jakarta	Hanoi	Phnom Penh
Bangkok	0	2	5	3	1	1	1
Manila	2	0	4	1	1	0	0
Singapore	5	4	0	3	4	2	3
Kuala Lumpur	3	1	3	0	0	3	3
Jakarta	1	1	4	0	0	0	0
Hanoi	1	0	2	3	0	0	0
Phnom Penh	1	0	3	3	0	0	0

a Draw a graph to represent the number of direct flights.
b Would this graph be considered to be directed or undirected? Why?
c In how many ways can you travel from:
 i Phnom Penh to Manila **ii** Hanoi to Bangkok?

5.3 Planar graphs

As indicated in Section 5.2, graphs can be drawn with intersecting edges. However, in many applications intersections may be undesirable. Consider a graph of an underground railway network. In this case intersecting edges would indicate the need for one rail line to be in a much deeper tunnel, which could add significantly to construction costs.

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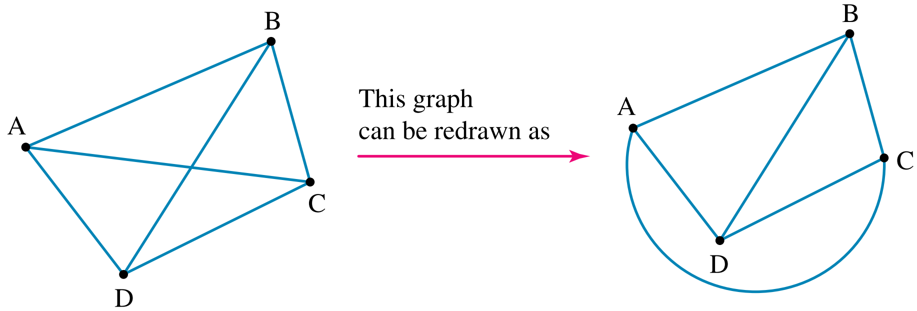
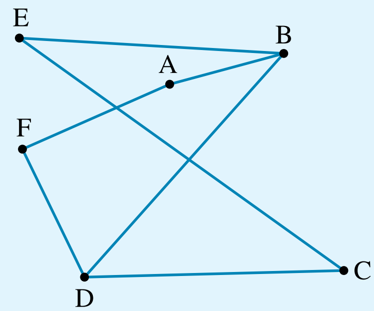
Interactivity
Planar graphs
int-6467

In some cases it is possible to redraw graphs so that they have no intersecting edges. When a graph can be redrawn in this way, it is known as a **planar graph**. For example, in the graph shown below, it is possible to redraw one of the intersecting edges so that it still represents the same information.



WORKED EXAMPLE 5

Redraw the graph so that it has no intersecting edges.



THINK

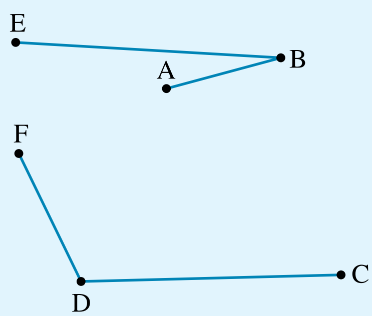
1 List all connections in the original graph.

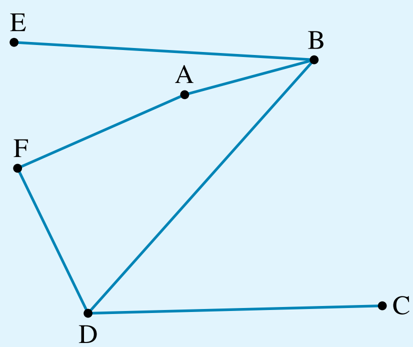
2 Draw all vertices and any section(s) of the graph that have no intersecting edges.

3 Draw any further edges that don't create intersections. Start with edges that have the fewest intersections in the original drawing.

WRITE/DRAW

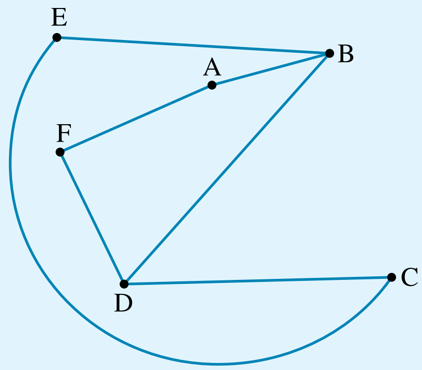
Connections: AB; AF; BD; BE; CD; CE; DF





4 Identify any edges yet to be drawn and redraw so that they do not intersect with the other edges.

Connections: ~~AB~~; ~~AF~~; ~~BD~~; ~~BE~~; ~~CD~~; CE; ~~DF~~



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Topic 2
Concept 3

Faces, vertices, edges and Euler's formula
Concept summary
Practice questions

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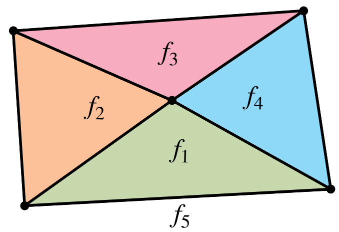
Interactivity
Euler's formula
int-6468

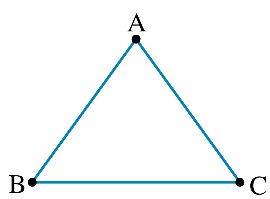
Euler's formula

In all planar graphs, the edges and vertices create distinct areas referred to as **faces**.

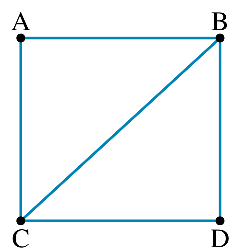
The planar graph shown in the diagram at right has five faces including the area around the outside.



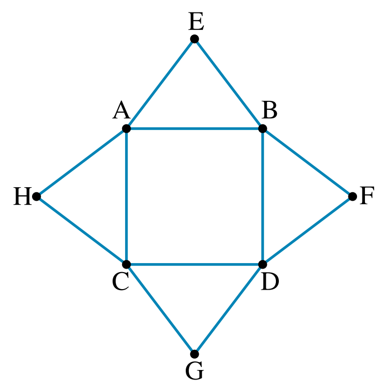
Consider the following group of planar graphs.



Graph 1



Graph 2



Graph 3

The number of vertices, edges and faces for each graph is summarised in the following table.

Graph	Vertices	Edges	Faces
Graph 1	3	3	2
Graph 2	4	5	3
Graph 3	8	12	6

For each of these graphs, we can obtain a result that is well known for any planar graph: the difference between the vertices and edges added to the number of faces will always equal 2.

Graph 1: $3 - 3 + 2 = 2$

Graph 2: $4 - 5 + 3 = 2$

Graph 3: $8 - 12 + 6 = 2$

This is known as Euler's formula for connected planar graphs and can be summarised as:

$v - e + f = 2$, where v is the number of vertices, e is the number of edges and f is the number of faces.

WORKED EXAMPLE 6

How many faces will there be for a connected planar graph of 7 vertices and 10 edges?

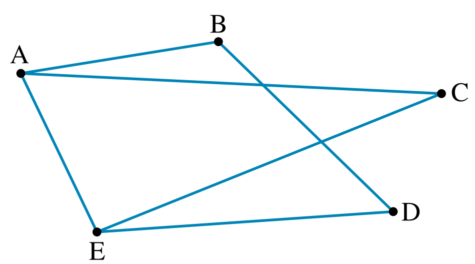
THINK	WRITE
1 Substitute the given values into Euler's formula.	$v - e + f = 2$ $7 - 10 + f = 2$
2 Solve the equation for the unknown value.	$7 - 10 + f = 2$ $f = 2 - 7 + 10$ $f = 5$
3 State the final answer.	There will be 5 faces in a connected planar graph with 7 vertices and 10 edges.

EXERCISE 5.3 Planar graphs

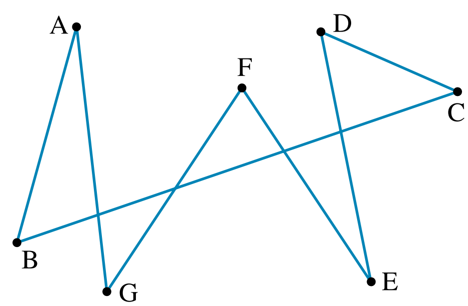
PRACTISE

1 WE5 Redraw the following graphs so that they have no intersecting edges.

a

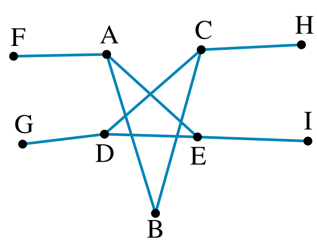


b

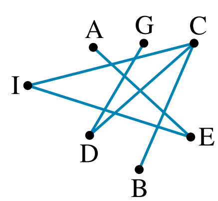


2 Which of the following are planar graphs?

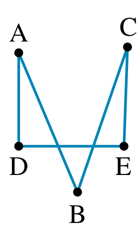
a A



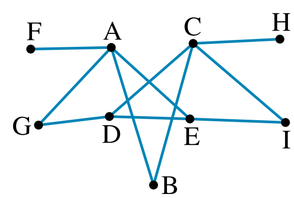
B



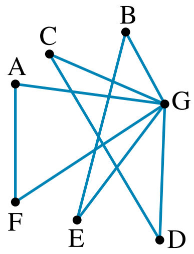
C



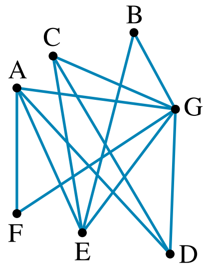
D



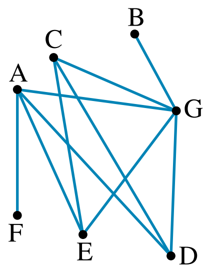
b **A**



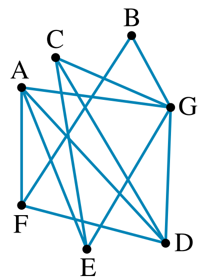
B



C



D



3 WE6 How many faces will there be for a connected planar graph of:

a 8 vertices and 10 edges

b 11 vertices and 14 edges?

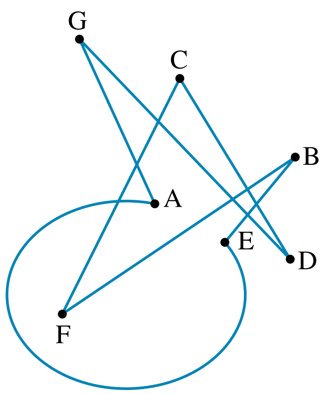
4 **a** For a connected planar graph of 5 vertices and 3 faces, how many edges will there be?

b For a connected planar graph of 8 edges and 5 faces, how many vertices will there be?

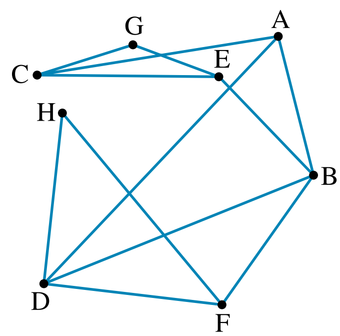
CONSOLIDATE

5 Redraw the following graphs to show that they are planar.

a

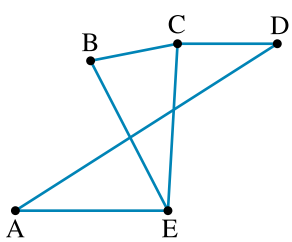


b

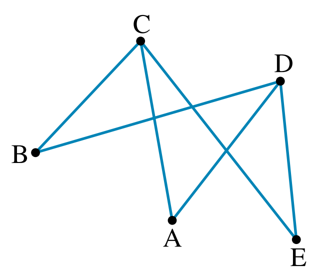


6 For each of the following planar graphs, identify the number of faces:

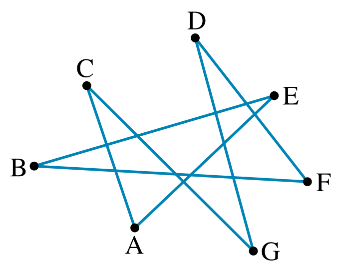
a



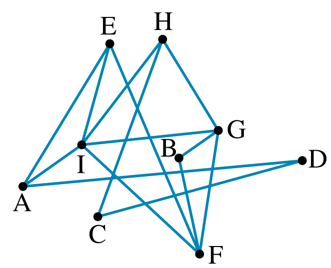
b



c



d



7 Construct a connected planar graph with:

a 6 vertices and 5 faces **b** 11 edges and 9 faces.

8 Use the following adjacency matrices to draw graphs that have no intersecting edges.

a $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

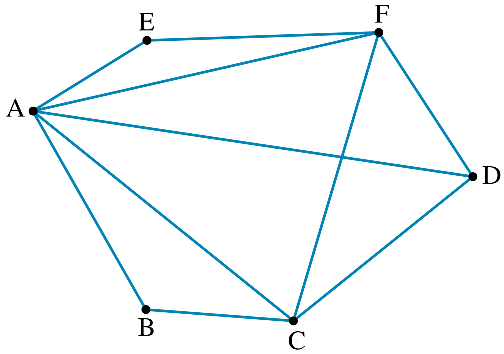
b $\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

9 For the graphs in question **8**:

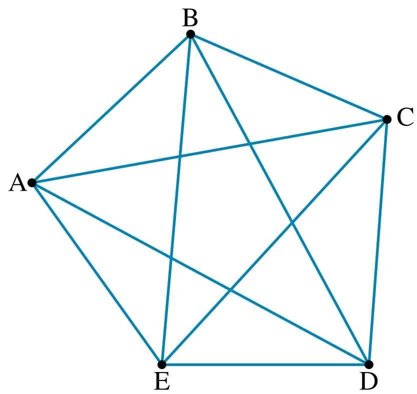
i identify the number of enclosed faces

ii identify the maximum number of additional edges that can be added to maintain a simple planar graph.

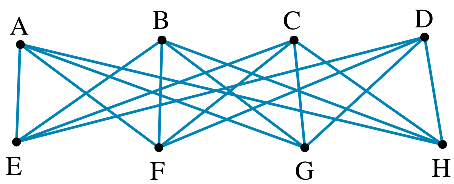
10 Which of the following graphs are not planar?



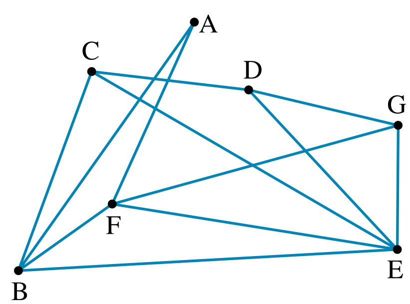
Graph 1



Graph 2

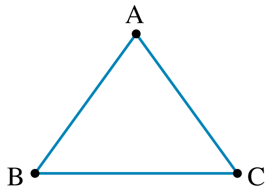


Graph 3

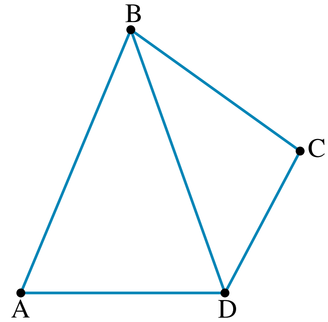


Graph 4

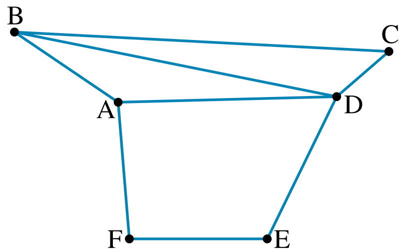
11 a Use the planar graphs shown to complete the table.



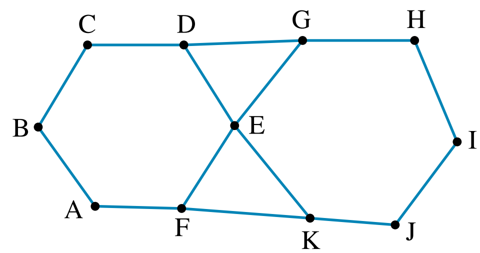
Graph 1



Graph 2



Graph 3

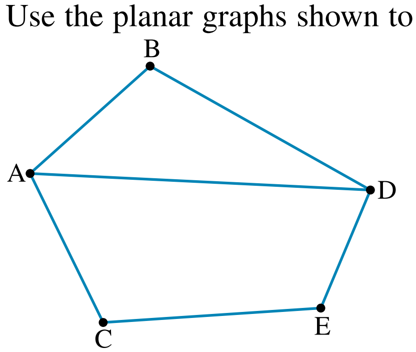


Graph 4

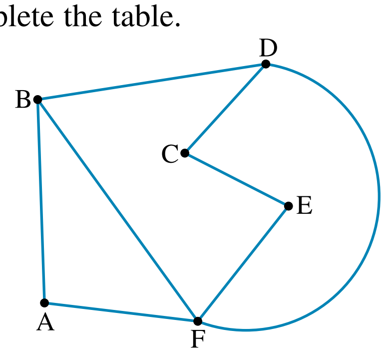
Graph	Total edges	Total degrees
Graph 1		
Graph 2		
Graph 3		
Graph 4		

b What pattern is evident from the table?

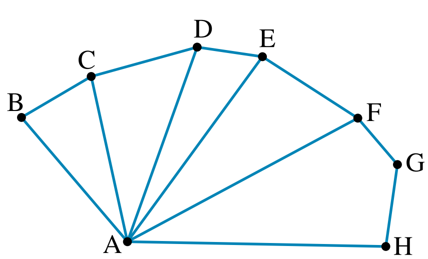
12 a Use the planar graphs shown to complete the table.



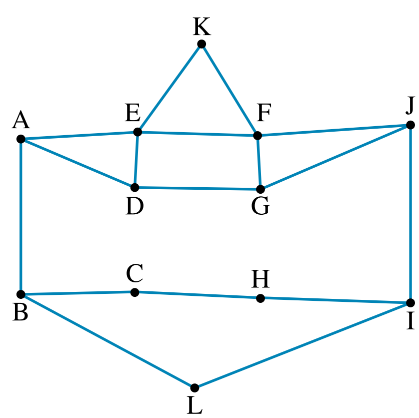
Graph 1



Graph 2



Graph 3

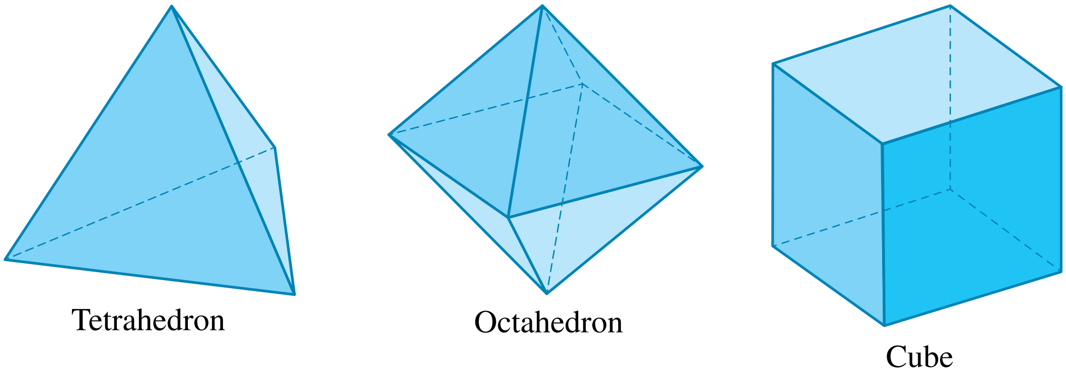


Graph 4

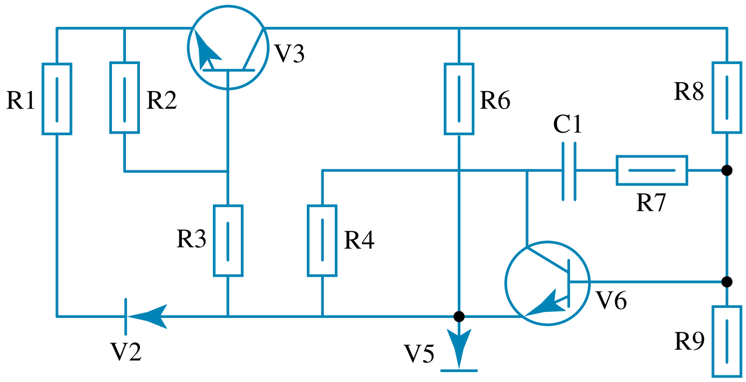
Graph	Total vertices of even degree	Total vertices of odd degree
Graph 1		
Graph 2		
Graph 3		
Graph 4		

b Is there any pattern evident from this table?

13 Represent the following 3-dimensional shapes as planar graphs.



14 A section of an electric circuit board is shown in the diagram.

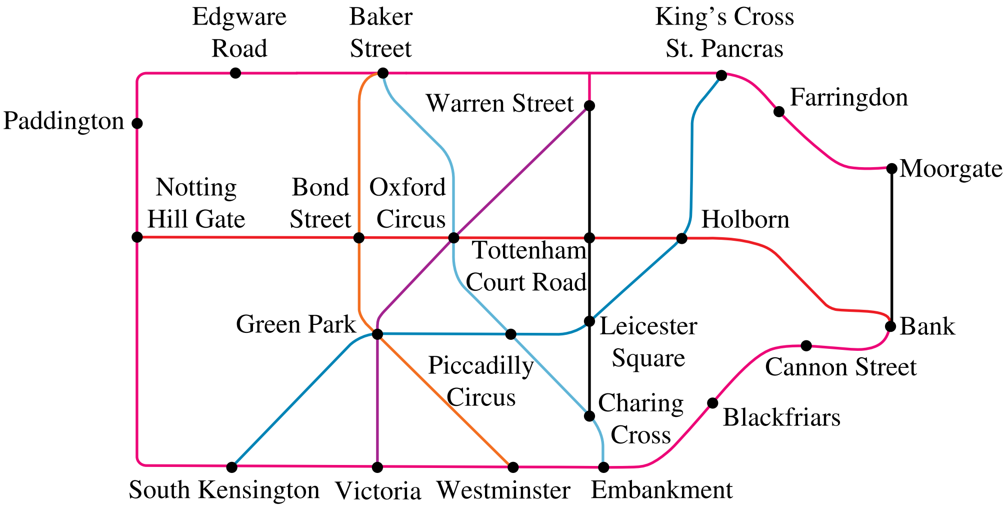


a Draw a graph to represent the circuit board, using vertices to represent the labelled parts of the diagram.

b Is it possible to represent the circuit board as a planar graph?

MASTER

15 The diagram shows a section of the London railway system.



a Display this information using an adjacency matrix.

b What does the sum of the rows of this adjacency matrix indicate?

16 The table displays the most common methods of communication for a group of people.

	Email	Facebook	SMS
Adam	Ethan, Liam	Ethan, Liam	Ethan
Michelle		Sophie, Emma, Ethan	Sophie, Emma
Liam	Adam		
Sophie		Michelle, Chloe	Michelle, Chloe
Emma	Chloe	Chloe, Ethan, Michelle	Chloe, Ethan
Ethan		Emma, Adam, Michelle	Emma
Chloe	Emma, Sophie	Emma, Sophie	Emma, Sophie

a Display the information for the entire table in a graph.
b Who would be the best person to introduce Chloe and Michelle?
c Display the Facebook information in a separate graph.
d If Liam and Sophie began communicating through Facebook, how many faces would the graph from part c then have?

5.4 Connected graphs

Traversing connected graphs

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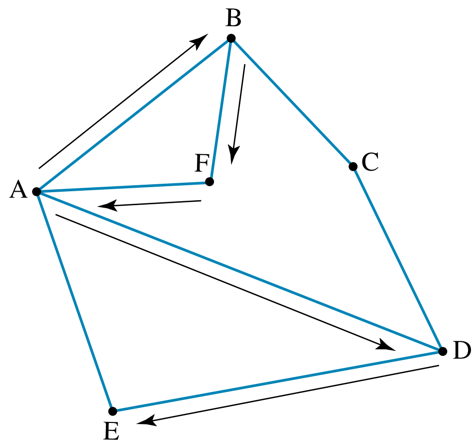
Interactivity
Traversing connected graphs
int-6469

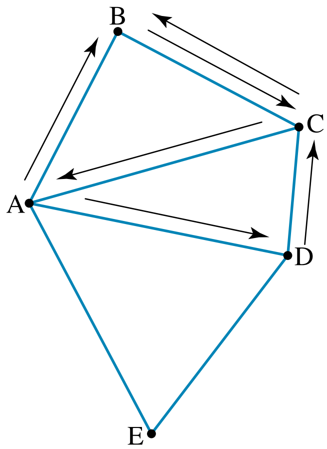
Many applications of graphs involve an analysis of movement around a network. These could include fields such as transport, communications or utilities, to name a few. Movement through a simple connected graph is described in terms of starting and finishing at specified vertices by travelling along the edges. This is usually done by listing the labels of the vertices visited in the correct order. In more complex graphs, edges may also have to be indicated, as there may be more than one connection between vertices.



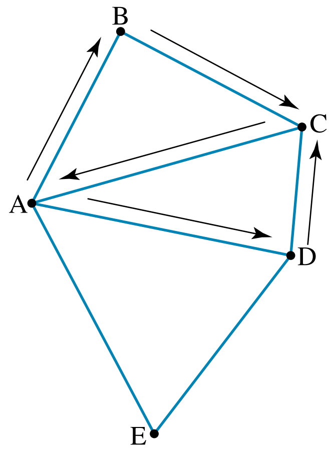
Route: ABFADE

The definitions of the main terms used when describing movement across a network are as follows.

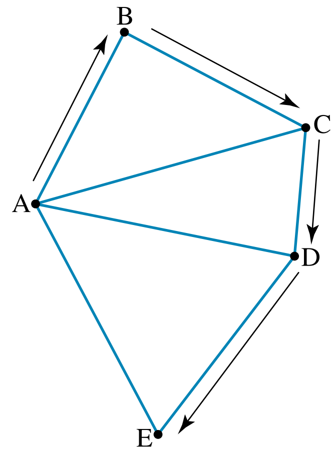
Walk: Any route taken through a network, including routes that repeat edges and vertices
Trail: A walk in which no edges are repeated
Path: A walk in which no vertices are repeated, except possibly the start and finish
Cycle: A path beginning and ending at the same vertex
Circuit: A trail beginning and ending at the same vertex



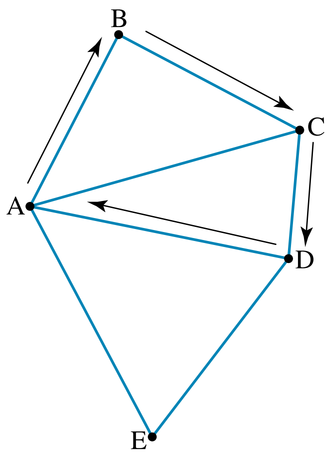
Walk: ABCADCB



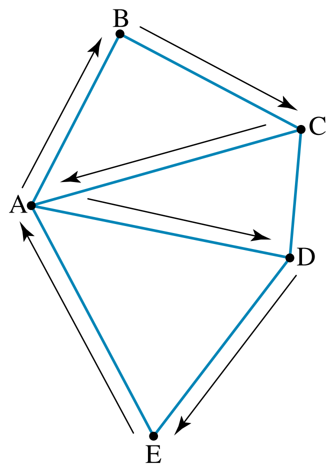
Trail: ABCADC



Path: ABCDE



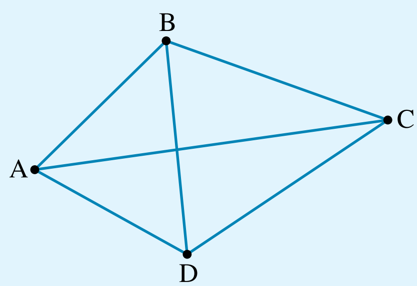
Cycle: ABCDA



Circuit: ABCADEA

WORKED EXAMPLE 7

In the following network, identify two different routes: one cycle and one circuit.



THINK	WRITE
1 For a cycle, identify a route that doesn't repeat a vertex apart from the start/finish.	Cycle: ABDCA
2 For a circuit, identify a route that doesn't repeat an edge and ends at the starting vertex.	Circuit: ADBCA

Euler trails and circuits

In some practical situations, it is most efficient if a route travels along each edge only once. Examples include parcel deliveries and council garbage collections. If it is possible to travel a network using each edge only once, the route is known as an **Euler trail** or **Euler circuit**.

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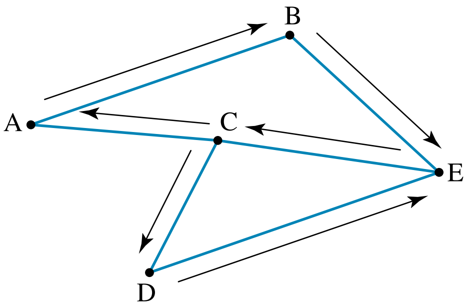
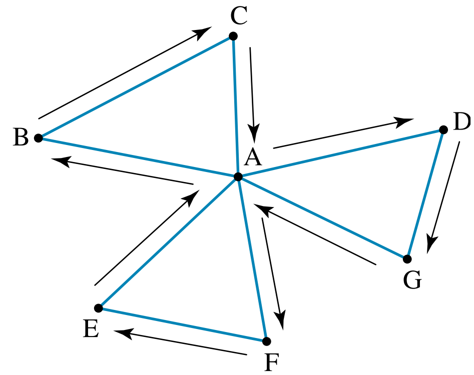
Euler trails and circuits
Concept summary
Practice questions

> **An Euler trail is a trail in which every edge is used once.**
> **An Euler circuit is a circuit in which every edge is used once.**



Euler trail: CDECABE



Euler circuit: ABCADGAFEA

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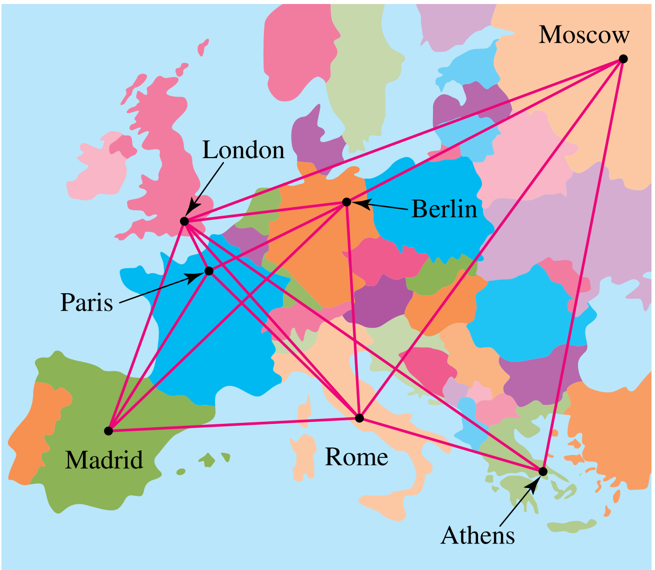
Interactivity
Euler trails and Hamiltonian paths
int-6470

Note that in the examples shown, the vertices for the Euler circuit are of even degree, and there are 2 vertices of odd degree for the Euler trail.

> **If all of the vertices of a connected graph are even, then an Euler circuit exists.**
> **If exactly 2 vertices of a connected graph are odd, then an Euler trail exists.**

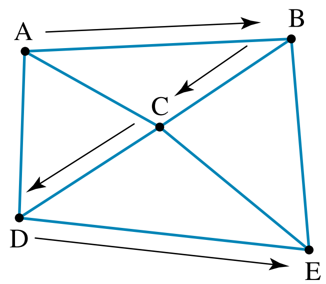
Hamiltonian paths and cycles

In other situations it may be more practical if all vertices can be reached without using all of the edges of the graph. For example, if you wanted to visit a selection of the capital cities of Europe, you wouldn't need to use all the available flight routes shown in the diagram.

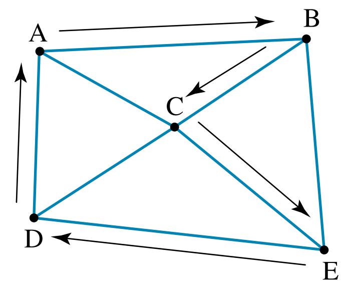


> A **Hamiltonian path** is a path that reaches all vertices of a network.
>
> A **Hamiltonian cycle** is a cycle that reaches all vertices of a network.

Hamiltonian paths and Hamiltonian cycles reach all vertices of a network once without necessarily using all of the available edges.



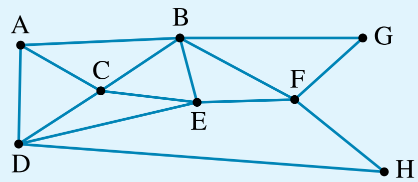
Hamiltonian path: ABCDE



Hamiltonian cycle: ABCEDA

WORKED EXAMPLE 8

Identify an Euler trail and a Hamiltonian path in the following graph.

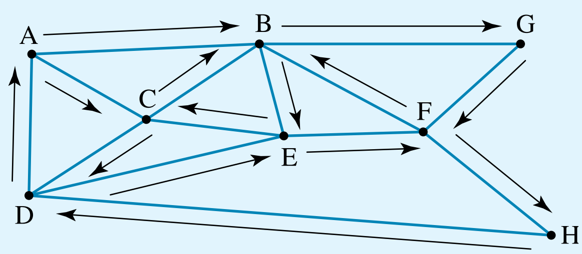


THINK | **WRITE/DRAW**

1 For an Euler trail to exist, there must be exactly 2 vertices with an odd-numbered degree.

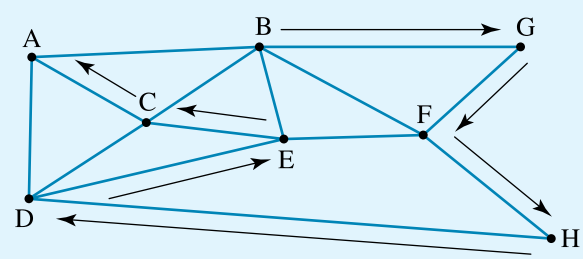
Deg(A) = 3, deg(B) = 5, deg(C) = 4, deg(D) = 4, deg(E) = 4, deg(F) = 4, deg(G) = 2, deg(H) = 2
As there are only two odd-degree vertices, an Euler trail must exist.

2 Identify a route that uses each edge once.



Euler trail: ABGFHDEFBECDACB

3 Identify a route that reaches each vertex once.



Hamiltonian path: BGFHDECA

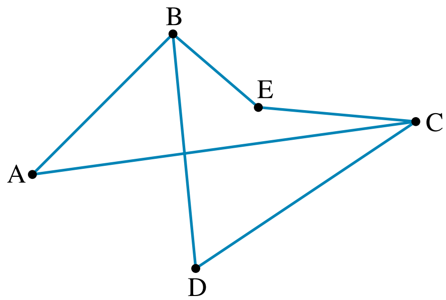
4 State the answer.

Euler trail: ABGFHDEFBECDACB
Hamiltonian path: BGFHDECA

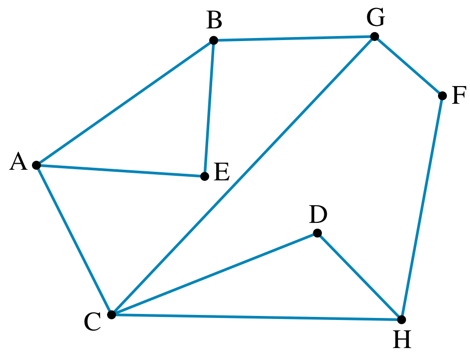
EXERCISE 5.4 Connected graphs

PRACTISE

1 WE7 In the following network, identify two different routes: one cycle and one circuit.

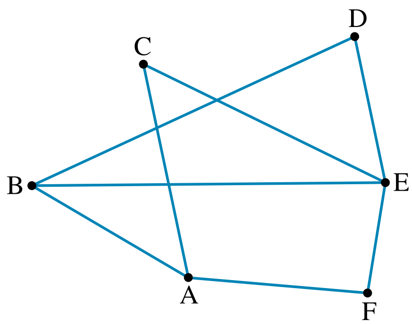


2 In the following network, identify three different routes: one path, one cycle and one circuit.

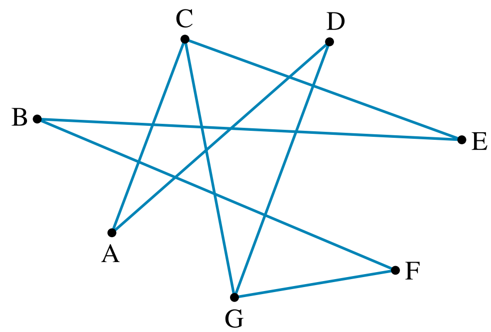


3 WE8 Identify an Euler trail and a Hamiltonian path in each of the following graphs.

a

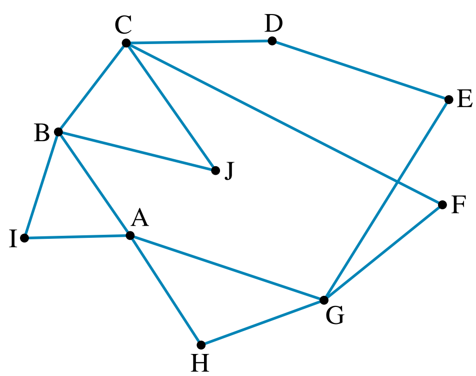


b

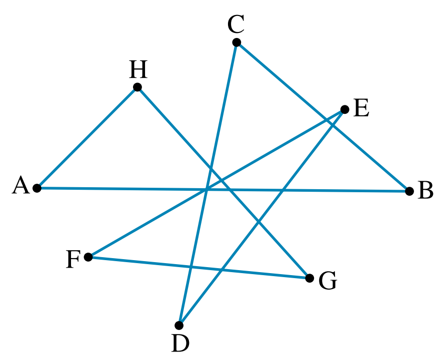


4 Identify an Euler circuit and a Hamiltonian cycle in each of the following graphs, if they exist.

a

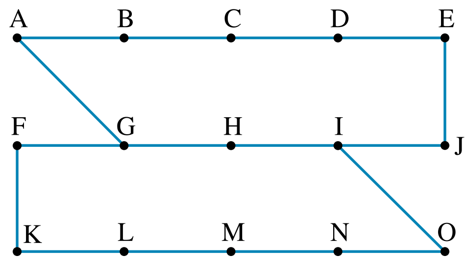


b



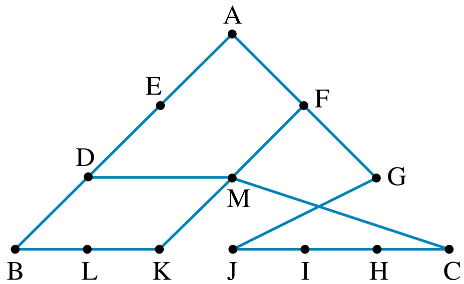
CONSOLIDATE

5 Which of the terms walk, trail, path, cycle and circuit could be used to describe the following routes on the graph shown?



a AGHIONMLKFGA
b IHGFKLMNO
c HIJEDCBAGH
d FGHIJEDCBAG

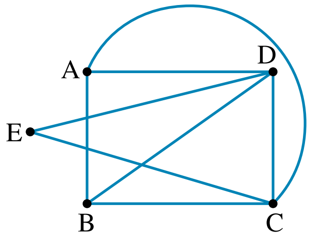
6 Use the following graph to identify the indicated routes.



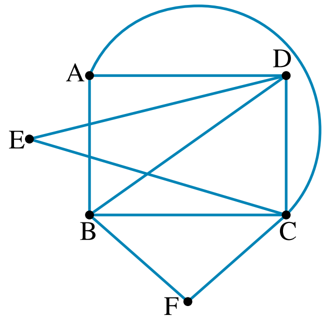
a A path commencing at M, including at least 10 vertices and finishing at D
b A trail from A to C that includes exactly 7 edges
c A cycle commencing at M that includes 10 edges
d A circuit commencing at F that includes 7 vertices

7 a Identify which of the following graphs have an Euler trail.

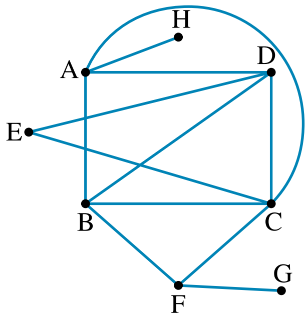
i



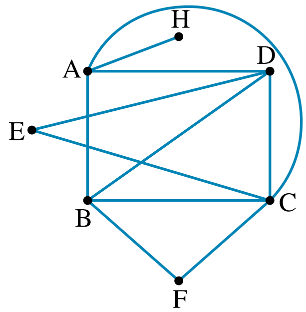
ii



iii



iv

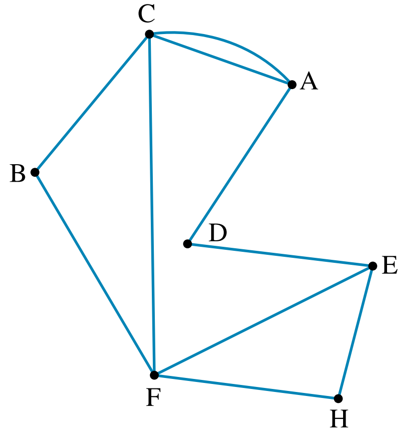


b Identify the Euler trails found.

8 a Identify which of the graphs from question 7 have a Hamiltonian cycle.
b Identify the Hamiltonian cycles found.

9 a Construct adjacency matrices for each of the graphs in question 7.
b How might these assist with making decisions about the existence of Euler trails and circuits, and Hamiltonian paths and cycles?

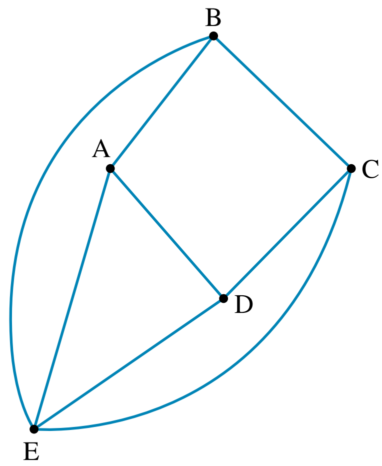
10 In the following graph, if an Euler trail commences at vertex A, at which vertices could it finish?



11 In the following graph, at which vertices could a Hamiltonian path finish if it commences by travelling from:

a B to E

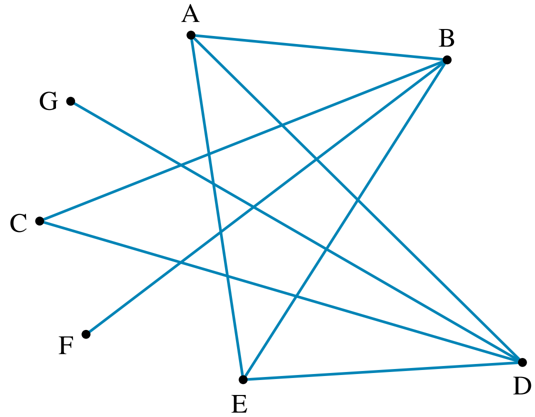
b E to A?



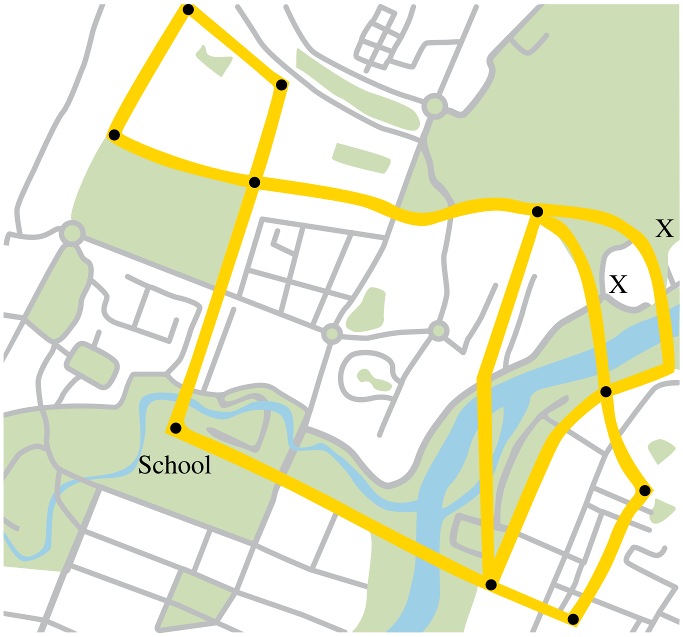
12 In the following graph, other than from G to F, between which 2 vertices must you add an edge in order to create a Hamiltonian path that commences from vertex:

a G

b F?



13 On the map shown, a school bus route is indicated in yellow. The bus route starts and ends at the school indicated.

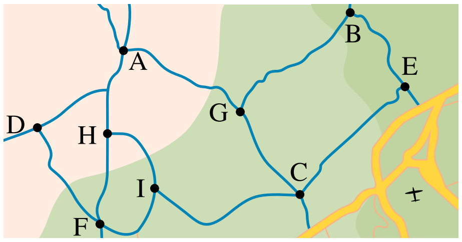


a Draw a graph to represent the bus route.

b Students can catch the bus at stops that are located at the intersections of the roads marked in yellow. Is it possible for the bus to collect students by driving down each section of the route only once? Explain your answer.

c If road works prevent the bus from travelling along the sections indicated by the Xs, will it be possible for the bus to still collect students on the remainder of the route by travelling each section only once? Explain your answer.

14 The map of an orienteering course is shown. Participants must travel to each of the nine checkpoints along any of the marked paths.



a Draw a graph to represent the possible ways of travelling to each checkpoint.

b What is the degree of checkpoint H?

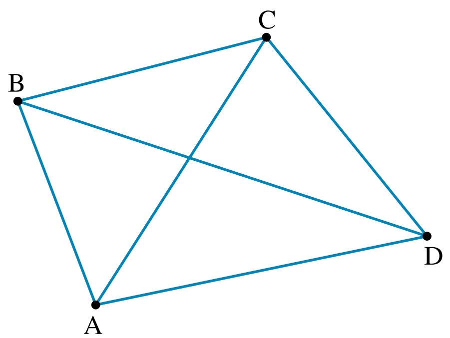
c If participants must start and finish at A and visit every other checkpoint only once, identify two possible routes they could take.

d **i** If participants can decide to start and finish at any checkpoint, and the paths connecting D and F, H and I, and A and G are no longer accessible, it is possible to travel the course by moving along each remaining path only once. Explain why.

ii Identify the two possible starting points.

MASTER

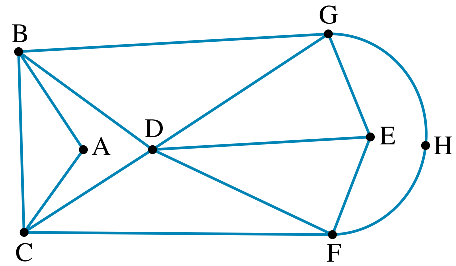
15 **a** Use the following complete graph to complete the table to identify all of the Hamiltonian cycles commencing at vertex A.



	Hamiltonian circuit
1.	ABCDA
2.	
3.	
4.	
5.	
6.	

b Are any other Hamiltonian cycles possible?

16 The graph shown outlines the possible ways a tourist bus can travel between eight locations.



a If vertex A represents the second location visited, list the possible starting points.
b If the bus also visited each location only once, which of the starting points listed in part a could not be correct?
c If the bus also needed to finish at vertex D, list the possible paths that could be taken.
d If instead the bus company decides to operate a route that travelled to each connection only once, what are the possible starting and finishing points?
e If instead the company wanted to travel to each connection only once and finish at the starting point, which edge of the graph would need to be removed?

5.5 Weighted graphs and trees

Weighted graphs

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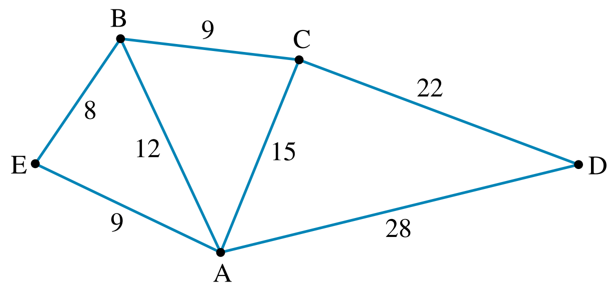
Units 1 & 2
AOS 3
Topic 2
Concept 5
Weighted graphs and minimum spanning trees
Concept summary
Practice questions

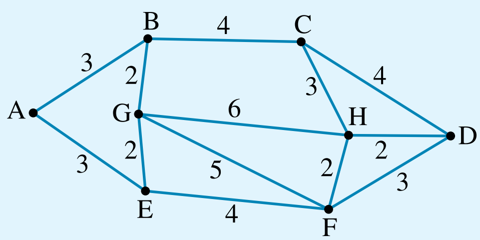
In many applications using graphs, it is useful to attach a value to the edges. These values could represent the length of the edge in terms of time or distance, or the costs involved with moving along that section of the path. Such graphs are known as **weighted graphs**.



Weighted graphs can be particularly useful as analysis tools. For example, they can help determine how to travel through a network in the shortest possible time.

WORKED EXAMPLE 9

The graph represents the distances in kilometres between eight locations.



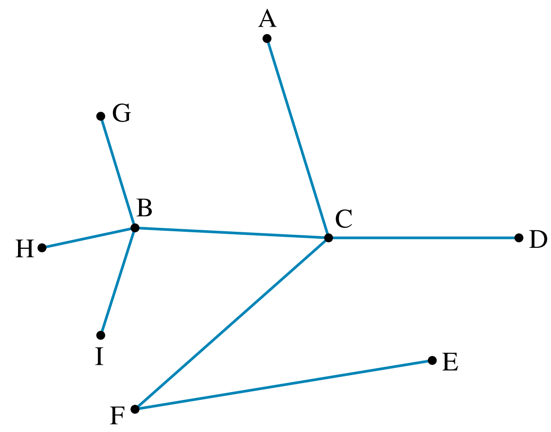
Identify the shortest distance to travel from A to D that goes to all vertices.

THINK	WRITE
1 Identify the Hamiltonian paths that connect the two vertices.	Possible paths: a ABGEFHCD b ABCHGEFD c AEGBCHFD d AEFGBCHD e AEFHGBCD

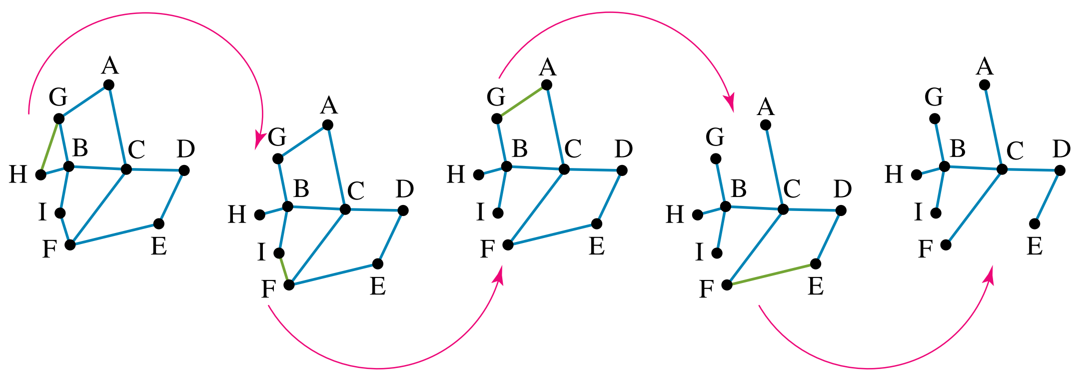
2 Calculate the total distances for each path to find the shortest.	**a** $3 + 2 + 2 + 4 + 2 + 3 + 4 = 20$ **b** $3 + 4 + 3 + 6 + 2 + 4 + 3 = 25$ **c** $3 + 2 + 2 + 4 + 3 + 2 + 3 = 19$ **d** $3 + 4 + 5 + 2 + 4 + 3 + 2 = 23$ **e** $3 + 4 + 2 + 6 + 2 + 4 + 4 = 25$
3 State the final answer.	The shortest distance from A to D that travels to all vertices is 19 km.

Trees

A **tree** is a simple connected graph with no circuits. As such, any pairs of vertices in a tree are connected by a unique path, and the number of edges is always 1 less than the number of vertices.



Spanning trees are sub-graphs (graphs that are formed from part of a larger graph) that include all of the vertices of the original graph. In practical settings, they can be very useful in analysing network connections. For example a *minimum* spanning tree for a weighted graph can identify the lowest-cost connections. Spanning trees can be obtained by systematically removing any edges that form a circuit, one at a time.



Prim's algorithm

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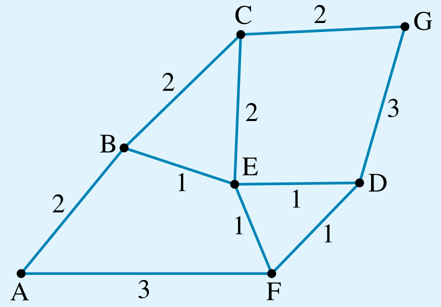
Interactivity
Minimum spanning tree and Prim's algorithm
int-6285

Prim's algorithm is a set of logical steps that can be used to identify the minimum spanning tree for a weighted connected graph.

Steps for Prim's algorithm:

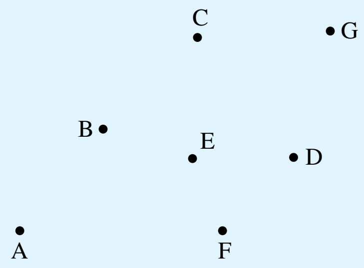
Step 1: Begin at a vertex with low weighted edges.
Step 2: Progressively select edges with the lowest weighting (unless they form a circuit).
Step 3: Continue until all vertices are selected.

WORKED EXAMPLE 10 Use Prim's algorithm to identify the minimum spanning tree of the graph shown.

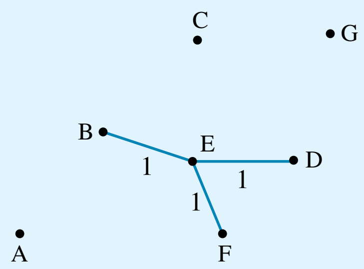


THINK / **DRAW**

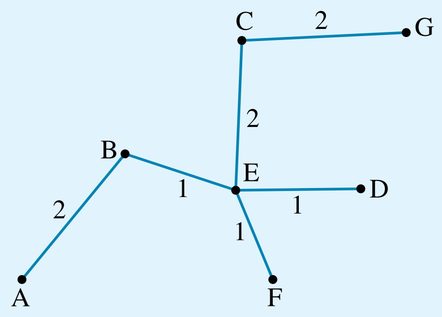
1 Draw the vertices of the graph.



2 Draw in any edges with the lowest weighting that do not complete a circuit.



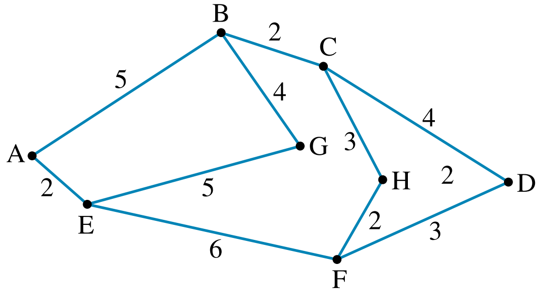
3 Draw in any edges with the next lowest weighting that do not complete a circuit. Continue until all vertices are connected.



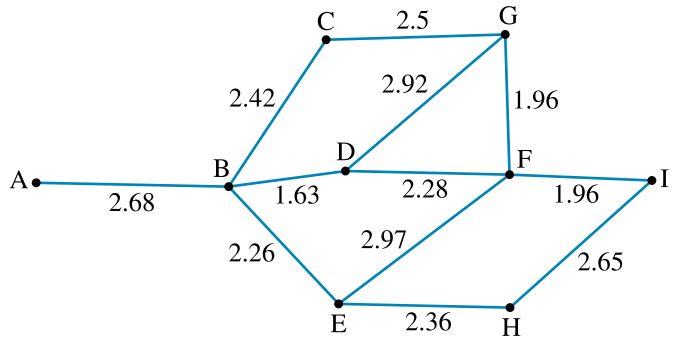
EXERCISE 5.5 Weighted graphs and trees

PRACTISE

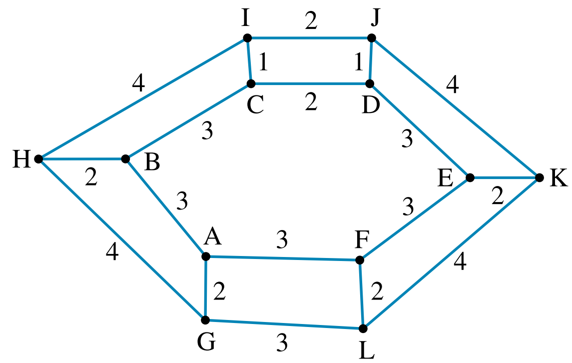
1 WE9 Use the graph to identify the shortest distance to travel from A to D that goes to all vertices.



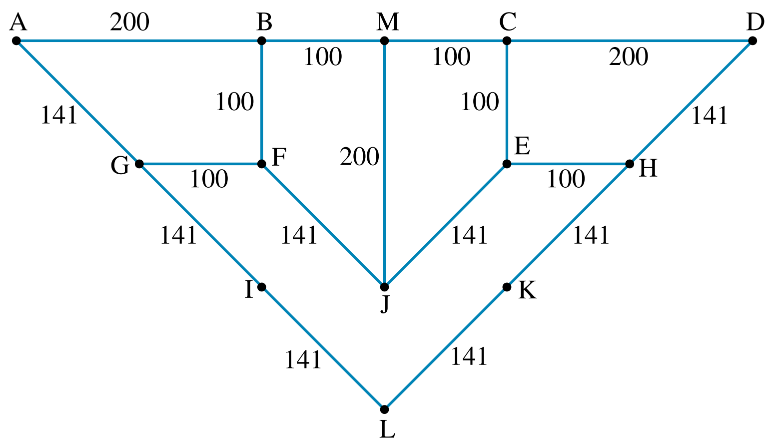
2 Use the graph to identify the shortest distance to travel from A to I that goes to all vertices.



3 WE10 Use Prim's algorithm to identify the minimum spanning tree of the graph shown.



4 Use Prim's algorithm to identify the minimum spanning tree of the graph shown.



CONSOLIDATE

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Units 1 & 2

AOS 3

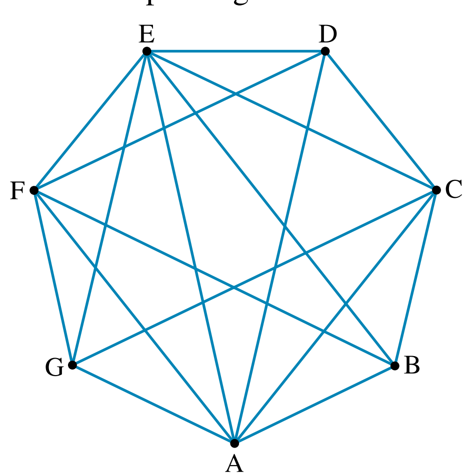
Topic 2

Concept 6

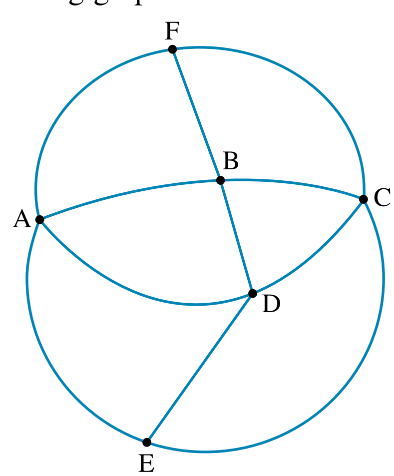
Prim's algorithm
Concept summary
Practice questions

5 Draw three spanning trees for each of the following graphs.

a



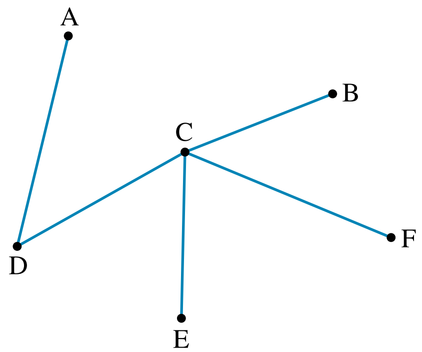
b



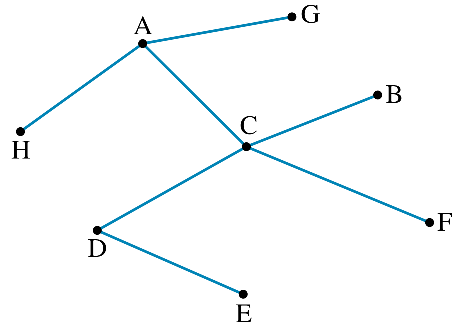
6 For the following trees:

i add the minimum number of edges to create an Euler trail
ii identify the Euler trail created.

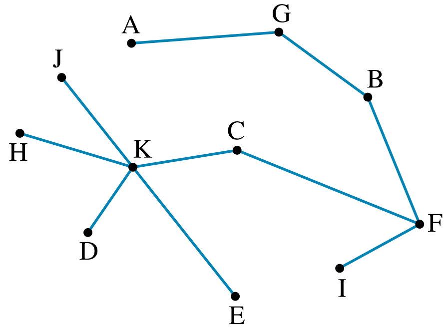
a



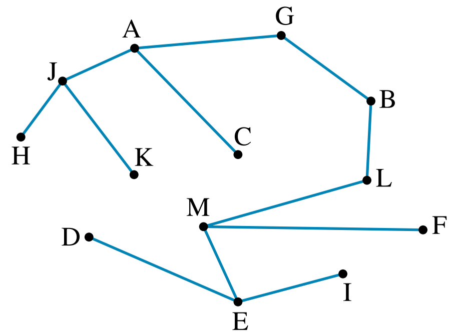
b



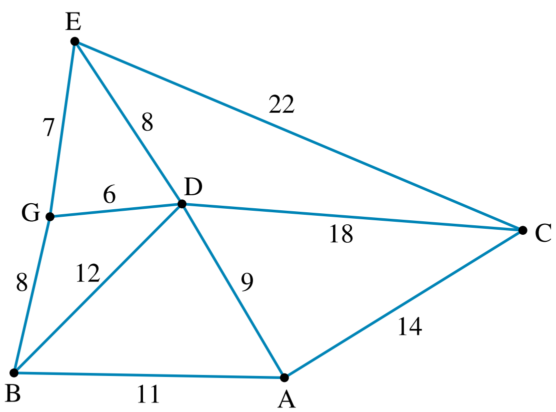
c



d



7 A truck starts from the main distribution point at vertex A and makes deliveries at each of the other vertices before returning to A. What is the shortest route the truck can take?

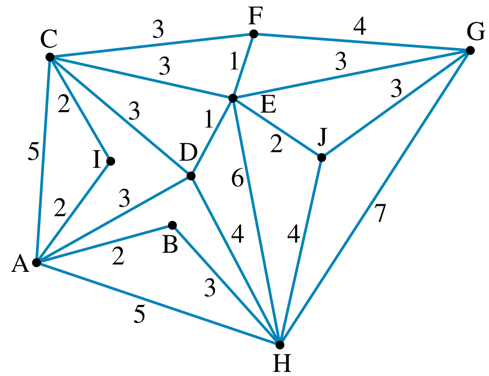


8 Part of the timetable and description for a bus route is shown in the table. Draw a weighted graph to represent the bus route.

Bus stop	Description	Time
Bus depot	The northernmost point on the route	7:00 am
Northsea Shopping Town	Reached by travelling south-east along a highway from the bus depot	7:15 am
Highview Railway Station	Travel directly south along the road from Northsea Shopping Town.	7:35 am
Highview Primary School	Directly east along a road from the railway station	7:40 am

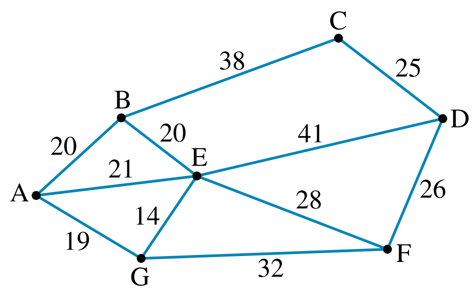
Bus stop	Description	Time
Eastend Medical Centre	Continue east along the road from the railway station.	7:55 am
Eastend Village	South-west along a road from the medical centre	8:05 am
Southpoint Hotel	Directly south along a road from Eastend Village	8:20 am
South Beach	Travel south-west along a road from the hotel.	8:30 am

9 Draw diagrams to show the steps you would follow when using Prim's algorithm to identify the minimum spanning tree for the following graph.

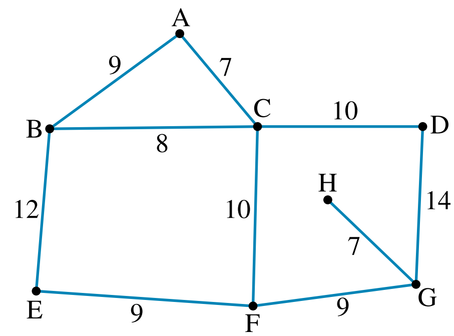


10 Identify the minimum spanning tree for each of the following graphs.

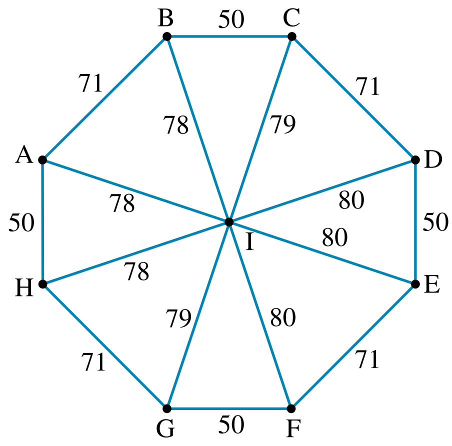
a



b



11 Consider the graph shown.

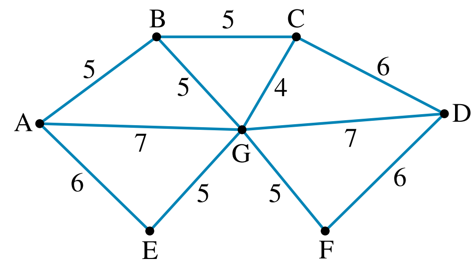


a Identify the longest and shortest Hamiltonian paths.
b What is the minimum spanning tree for this graph?

12 Consider the graph shown.

a If an edge with the highest weighting is removed, identify the shortest Hamiltonian path.

b If the edge with the lowest weighting is removed, identify the shortest Hamiltonian path.

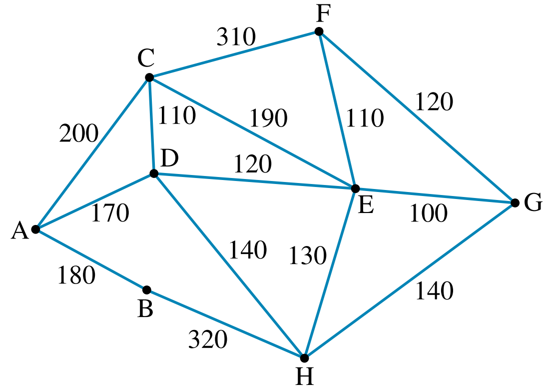


13 The weighted graph represents the costs incurred by a salesman when moving between the locations of various businesses.

a What is the cheapest way of travelling from A to G?

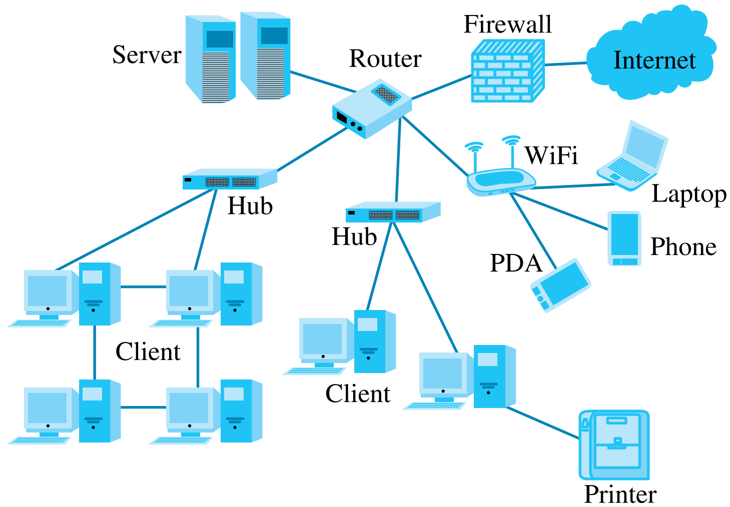
b What is the cheapest way of travelling from B to G?

c If the salesman starts and finishes at E, what is the cheapest way to travel to all vertices?

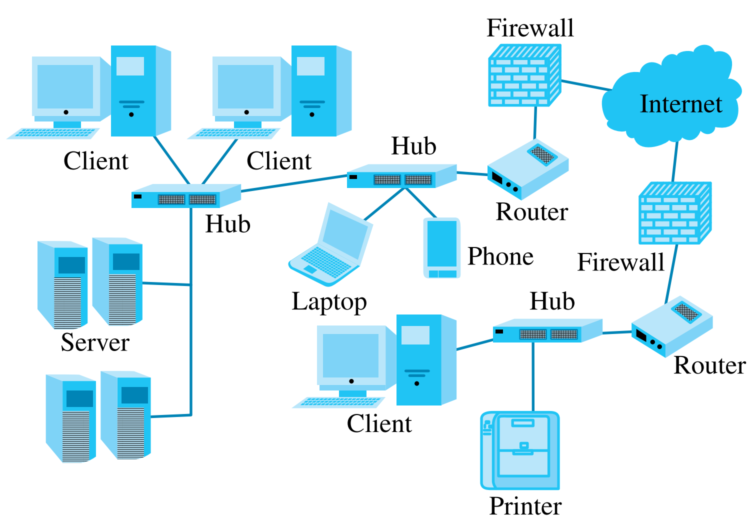


14 The diagrams show two options for the design of a computer network for a small business.

Option 1



Option 2



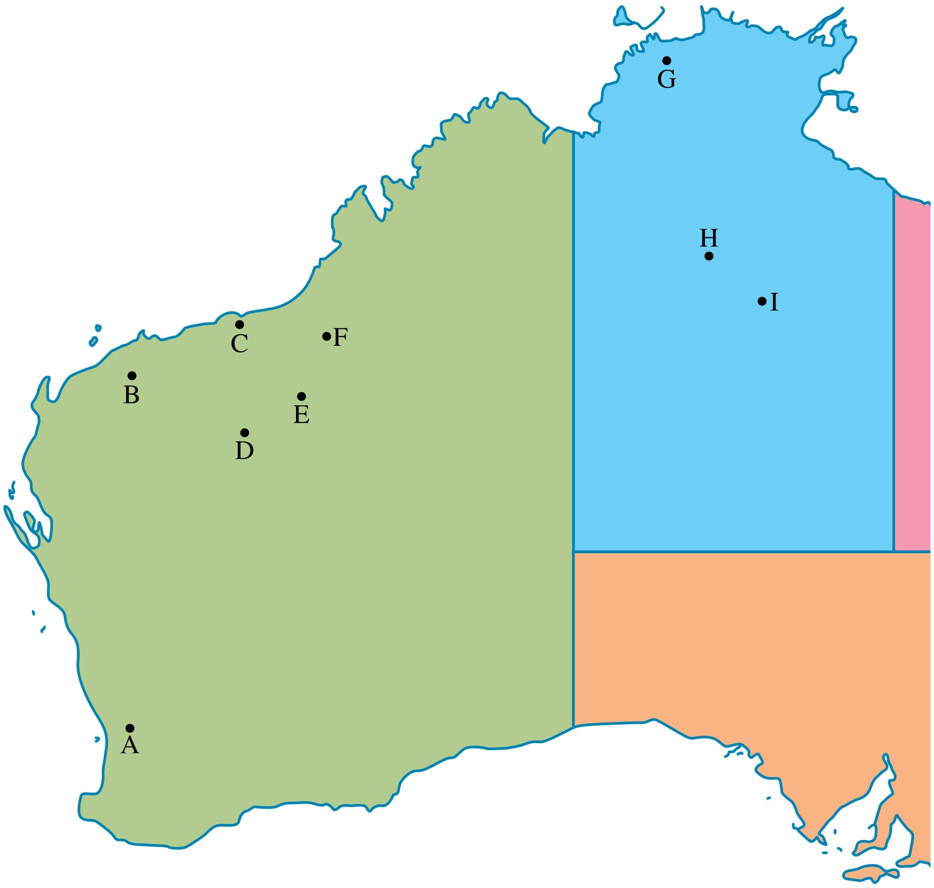
Information relating to the total costs of setting up the network is shown in the following table.

Connected to:	Server	Client	Hub	Router	Firewall	Wifi	Printer
Server			$995	$1050			
Client		$845	$355				$325
Hub			$365	$395			$395
Router	$1050		$395		$395	$395	
Laptop			$295			$325	
Phone			$295			$325	
PDA						$325	
Internet					$855		

a Use this information to draw a weighted graph for each option.
b Which is the cheapest option?

MASTER

15 A mining company operates in several locations in Western Australia and the Northern Territory, as shown on the map.

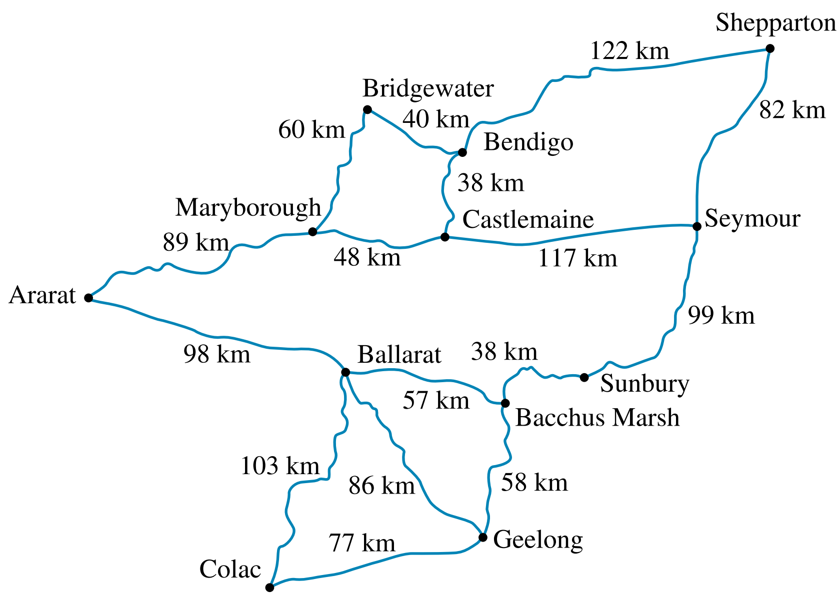


Flights operate between selected locations, and the flight distances (in km) are shown in the following table.

	A	B	C	D	E	F	G	H	I
A		1090		960			2600		2200
B	1090		360	375	435				
C		360							
D	960	375							
E		435							
F							1590	1400	
G	2600					1590		730	
H						1400	730		220
I	2200							220	

a Show this information as a weighted graph.
b Does a Hamiltonian path exist? Explain your answer.
c Identify the shortest distance possible for travelling to all sites the minimum number of times if you start and finish at:
  i A ii G.
d Draw the minimum spanning tree for the graph.

16 The organisers of the 'Tour de Vic' bicycle race are using the following map to plan the event.



a Draw a weighted graph to represent the map.
b If they wish to start and finish in Geelong, what is the shortest route that can be taken that includes a total of nine other locations exactly once, two of which must be Ballarat and Bendigo?
c Draw the minimal spanning tree for the graph.
d If the organisers decide to use the minimum spanning tree as the course, what would the shortest possible distance be if each location had to be reached at least once?

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5.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

To access eBookPLUS activities, log on to

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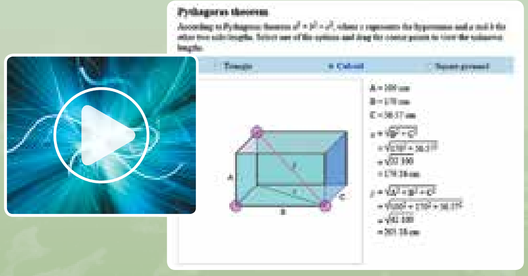
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.



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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

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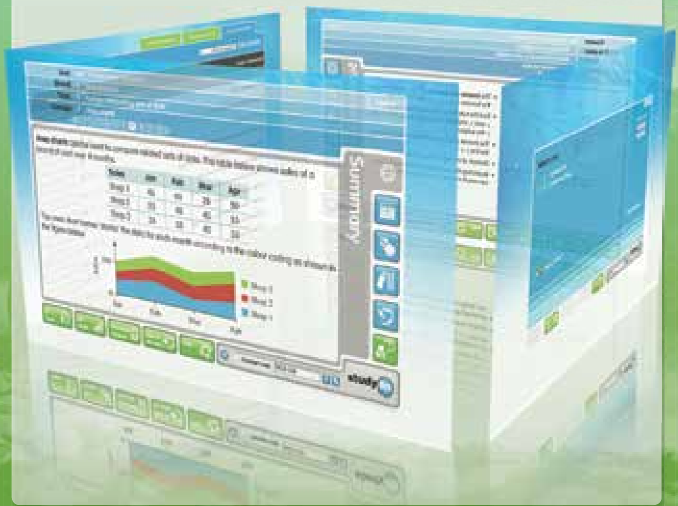
Units 1 & 2 **Graphs and networks**

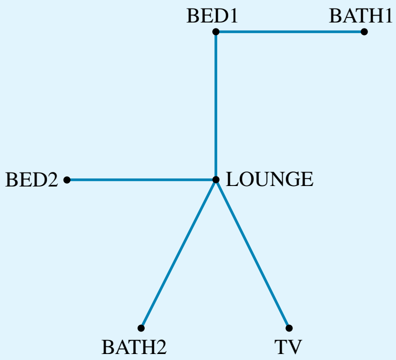
Sit topic test



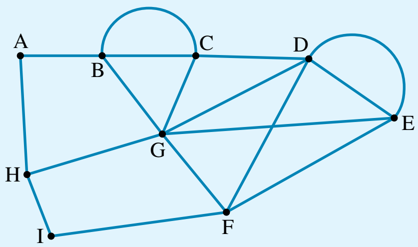
5 Answers

EXERCISE 5.2

1



2



3 a Edges = 7; Degree sum = 14

b Edges = 10; Degree sum = 20

4 a Edges = 9; Degree sum = 18

b Edges = 9; Degree sum = 18

5 a The graphs are isomorphic.

b The graphs are isomorphic.

c The graphs are not isomorphic.

d The graphs are isomorphic.

6 a Different degrees and connections

b Different connections

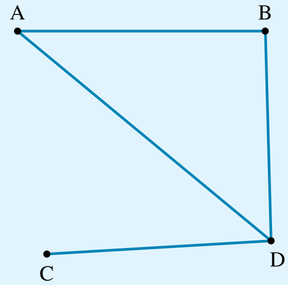
7 a $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

b $\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

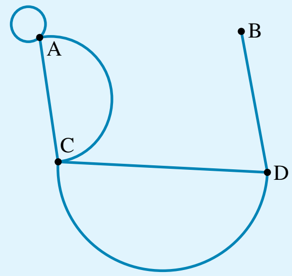
c $\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \end{bmatrix}$

d $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 4 & 0 \end{bmatrix}$

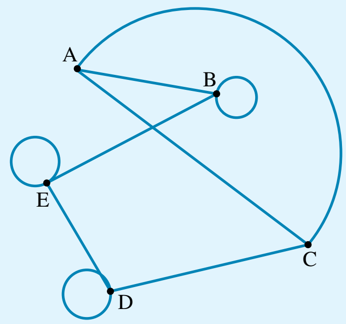
8 a



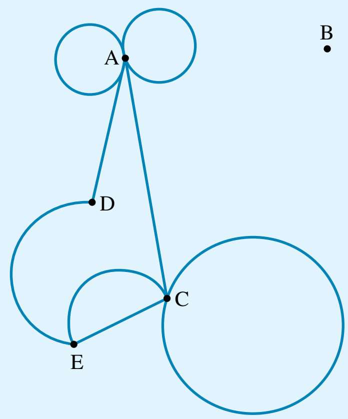
b



c



d



9 a $\deg(A) = 5$; $\deg(B) = 3$; $\deg(C) = 4$; $\deg(D) = 1$; $\deg(E) = 1$

b $\deg(A) = 0$; $\deg(B) = 2$; $\deg(C) = 2$; $\deg(D) = 3$; $\deg(E) = 3$

c $\deg(A) = 4$; $\deg(B) = 2$; $\deg(C) = 2$; $\deg(D) = 2$; $\deg(E) = 4$

d $\deg(A) = 1$; $\deg(B) = 2$; $\deg(C) = 1$; $\deg(D) = 1$; $\deg(E) = 3$

10

Graph	Simple	Complete	Connected
Graph 1	Yes	No	Yes
Graph 2	Yes	No	Yes
Graph 3	Yes	No	Yes
Graph 4	No	No	Yes
Graph 5	No	Yes	Yes

11 Graph 1:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Graph 2:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Graph 3:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Graph 4:

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Graph 5:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

12 The isomorphic pairs are graphs 2 and 4, and graphs 5 and 6.

13

Vertices	Edges
2	1
3	3
4	6
5	10
6	15
n	$\dfrac{n(n-1)}{2}$

14 **a**

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

b

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

c

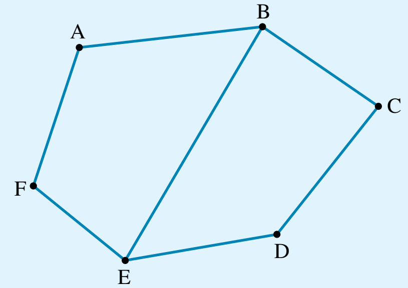
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$$

d

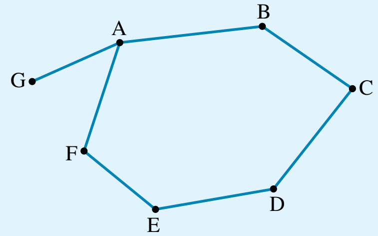
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

15 Answers will vary. Possible answers are shown.

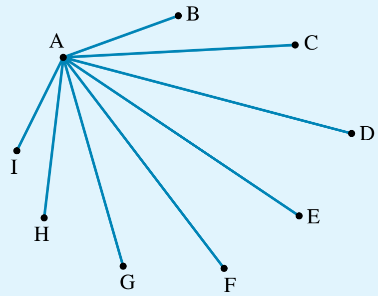
a



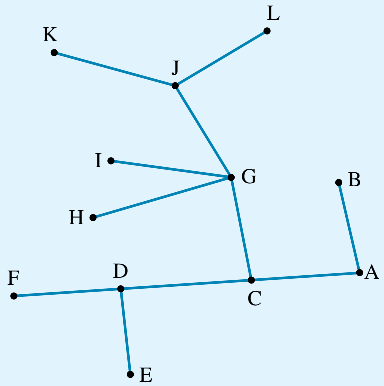
b



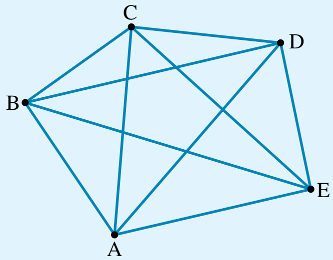
c



16



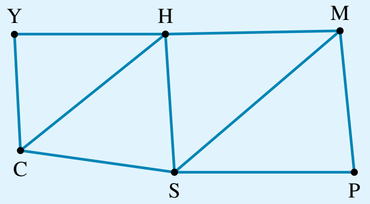
17 **a**



b Complete graph

c Total number of games played

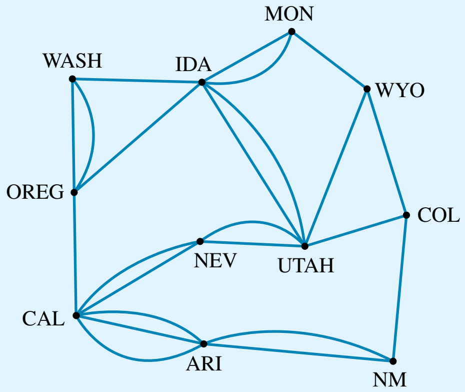
18 a



b Huairou and Shunyi

c Simple connected graph

19 a



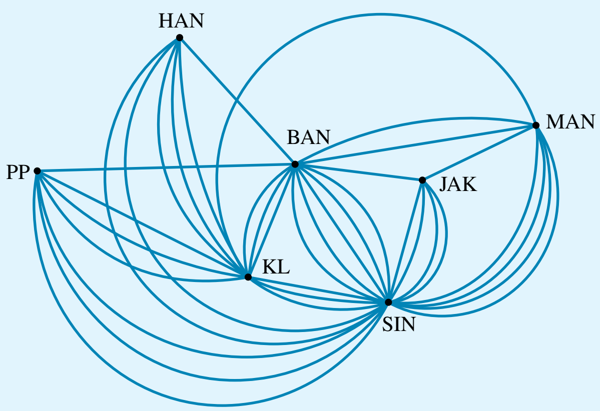
b

	Wa	O	Ca	I	N	A	M	U	Wy	Co	NM
Wa	0	2	0	1	0	0	0	0	0	0	0
O	2	0	1	1	0	0	0	0	0	0	0
Ca	0	1	0	0	2	3	0	0	0	0	0
I	1	1	0	0	0	0	2	2	0	0	0
N	0	0	2	0	0	0	0	2	0	0	0
A	0	0	3	0	0	0	0	0	0	0	2
M	0	0	0	2	0	0	0	0	1	0	0
U	0	0	0	2	2	0	0	0	1	1	0
Wy	0	0	0	0	0	0	1	1	0	1	0
Co	0	0	0	0	0	0	0	1	1	0	1
NM	0	0	0	0	0	2	0	0	0	1	0

c California, Idaho and Utah

d Montana

20 a



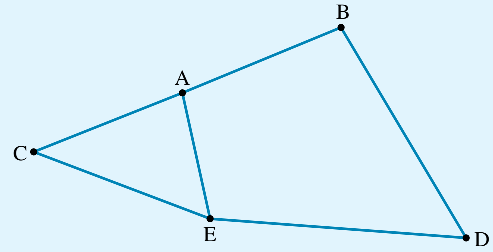
b Directed, as it would be important to know the direction of the flight

i 10

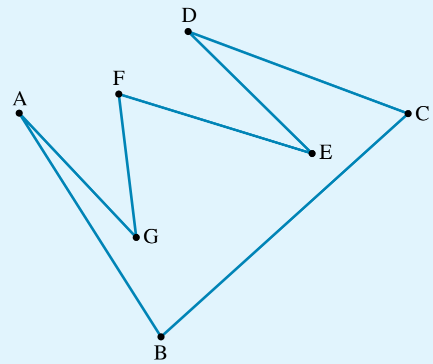
ii 7

EXERCISE 5.3

1 a



b



2 a All of them

b All of them

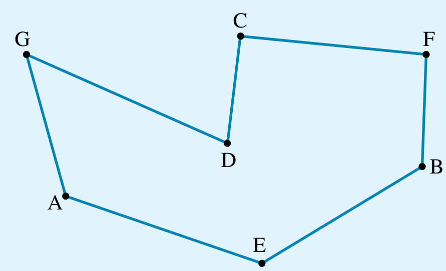
3 a 4

b 5

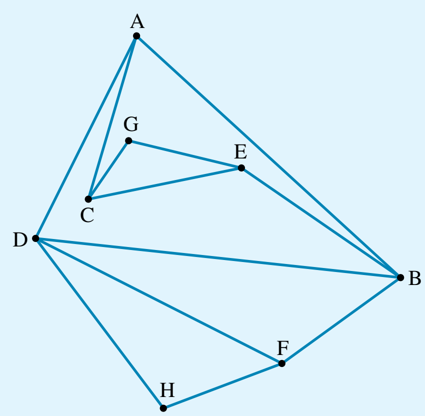
4 a 6

b 5

5 a



b



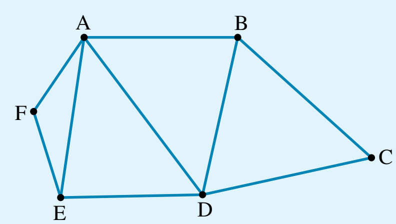
6 a 3

b 3

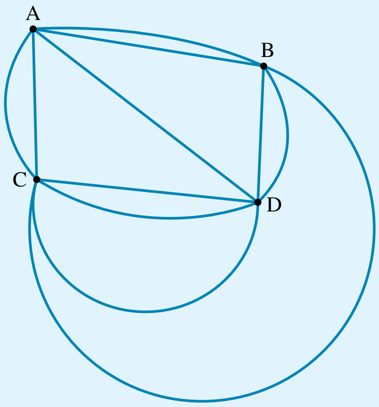
c 2

d 7

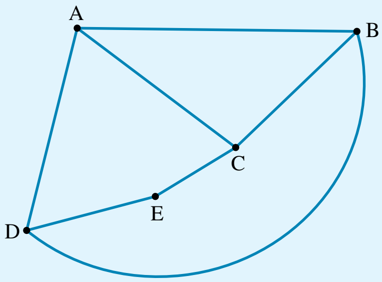
7 a



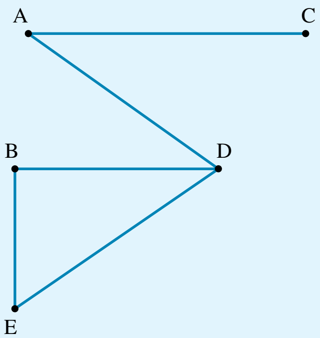
b



8 a



b



9 a **i** 3

ii 2

b **i** 1

ii 4

10 Graph 3

11 a

Graph	Total edges	Total degrees
Graph 1	3	6
Graph 2	5	10
Graph 3	8	16
Graph 4	14	28

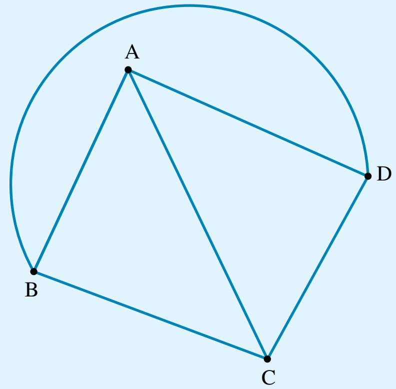
b Total degrees = 2 × total edges

12 a

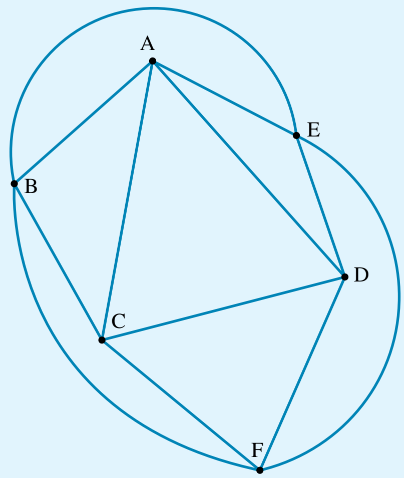
Graph	Total vertices of even degree	Total vertices of odd degree
Graph 1	3	2
Graph 2	4	2
Graph 3	4	6
Graph 4	6	6

b No clear pattern evident.

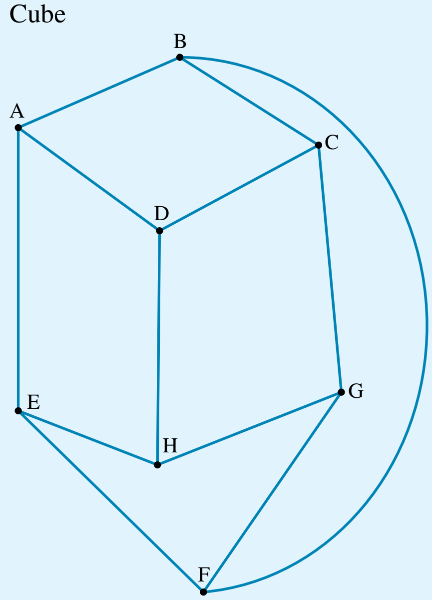
13 Tetrahedron



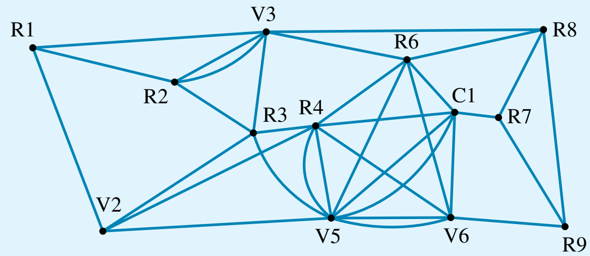
Octahedron



Cube



14 a

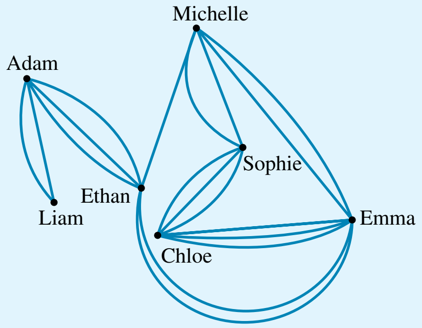


b No

15 a See the matrix at the foot of the page.*

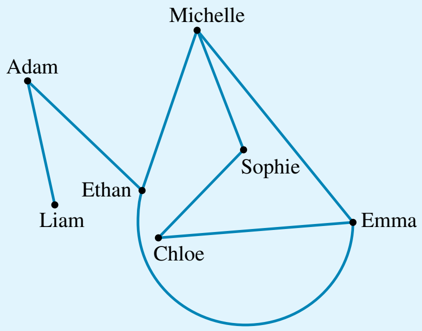
b The sum of the rows represents the sum of the degree of the vertices, or twice the number of edges (connections).

16 a



b Sophie or Emma

c



d 4

EXERCISE 5.4

1 Cycle: ABECA (others exist)

Circuit: BECDA (others exist)

2 Path: ABGFHDC (others exist)

Cycle: DCGFHD (others exist)

Circuit: AEBGFHDCA (others exist)

3 a Euler trail: AFEDBECAB; Hamiltonian path: BDECAF

b Euler trail: GFBECGDAC; Hamiltonian path: BECADGF

4 a Euler circuit: AIBAHGFCJBCDEGA; Hamiltonian cycle: none exist

b Euler circuit: ABCDEFGHA (others exist); Hamiltonian cycle: HABCDEFGH (others exist)

5 a Walk

b Walk, trail and path

c Walk, trail, path, cycle and circuit

d Walk and trail

6 a MCHIJGFAED

b AEDBLKMC

c MDEAFGJIHCM

d FMCHIJGF

7 a Graphs **i**, **ii** and **iv**

b Graph **i**: ACDABDECB (others exist)

*15a

	Pa	Ed	Bak	Wa	Ki	Fa	Mo	No	Bo	Ox	To	Ho	Ba	So	Vi	Gr	Pi	We	Em	Bl	Ca	Le	Ch
Pa	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Ed	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Bak	0	1	0	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
Wa	0	0	1	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
Ki	0	0	1	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
Fa	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Mo	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
No	1	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
Bo	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
Ox	0	0	1	1	0	0	0	0	1	0	1	0	0	0	0	1	1	0	0	0	0	0	0
To	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0
Ho	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0
Ba	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0
So	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
Vi	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0
Gr	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	1	1	0	0	0	0	0
Pi	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1
We	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0
Em	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1
Bl	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
Ca	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
Le	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	1
Ch	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0

Graph **ii**: CFBCEDBADCA (others exist)

Graph **iv**: CFBCEDCADBAH (others exist)

8 a Graphs **i** and **ii**

b Graph **i**: CEDABC

Graph **ii**: CEDABGC

9 a **i** $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

ii $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$

iii $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

iv $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

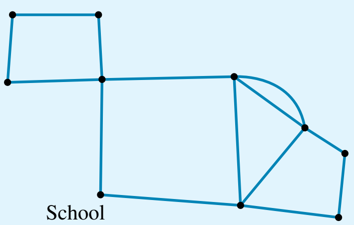
b The presence of Euler trails and circuits can be identified by using the adjacency matrix to check the degree of the vertices. The presence of Hamiltonian paths and cycles can be identified by using the adjacency matrix to check the connections between vertices.

10 E

11 a A or C **b** B or D

12 a G to C **b** F to E

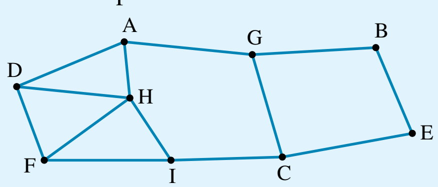
13 a



b Yes, because the degree of each intersection or corner point is an even number.

c Yes, because the degree of each remaining intersection or corner point is still an even number.

14 a



b 4

c **i** ADHFICEBGA

ii AHDFICEBGA

d **i** Yes, because two of the checkpoints have odd degree.

ii H and C

15 a

	Hamiltonian cycle
1.	ABCDA
2.	ABDCA
3.	ACBDA
4.	ACDBA
5.	ADBCA
6.	ADCBA

b Yes, commencing on vertices other than A

16 a B, C, D, F or G

b B or C

c None possible

d D or E

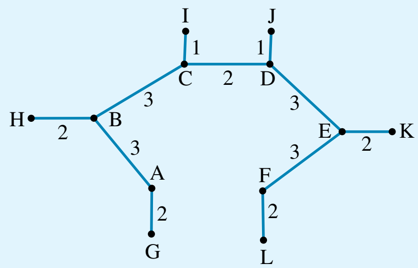
e D to E

EXERCISE 5.5

1 21

2 20.78

3



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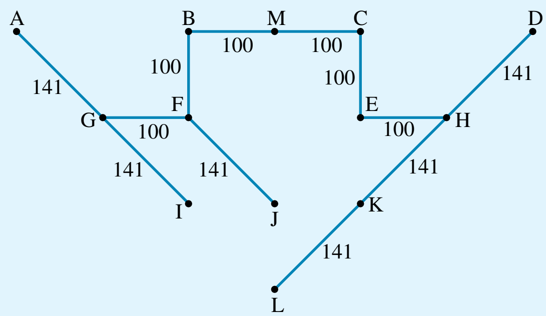
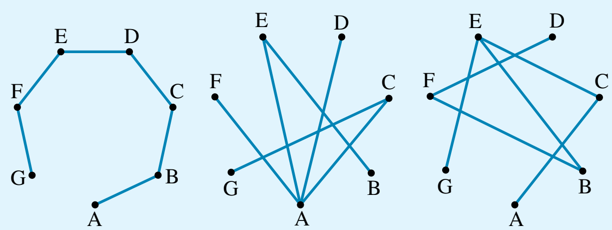
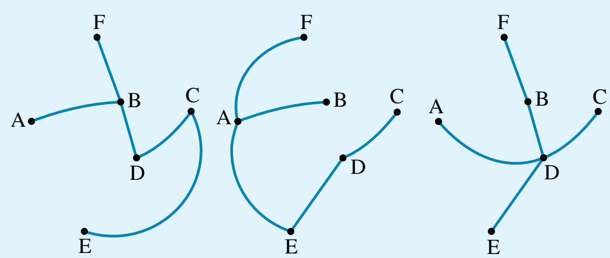


5 a



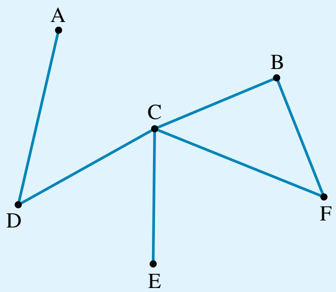
Other possibilities exist.

b



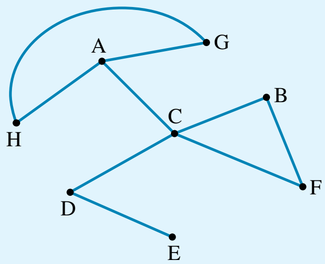
Other possibilities exist.

6 a i



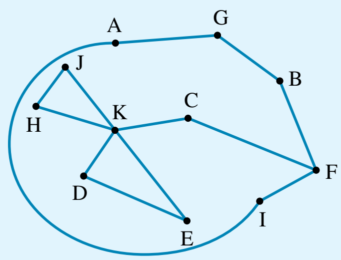
ii ADCBFCE or ADCFBCE

b i



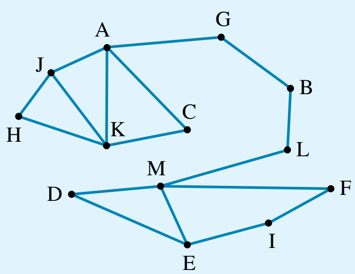
ii AHGACBFCDE or similar

c i



ii KDEKHJKCFIAGBF or similar

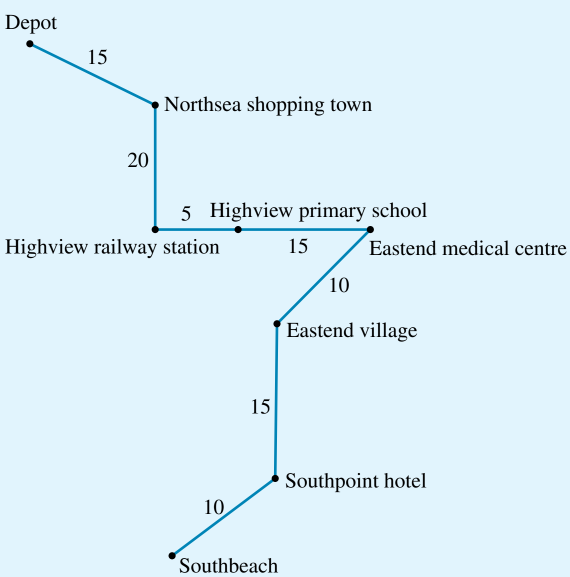
d i



ii EDMEIFMLBGACKHJKA or similar

7 ABGEDCA or ACDEGBA (length 66)

8



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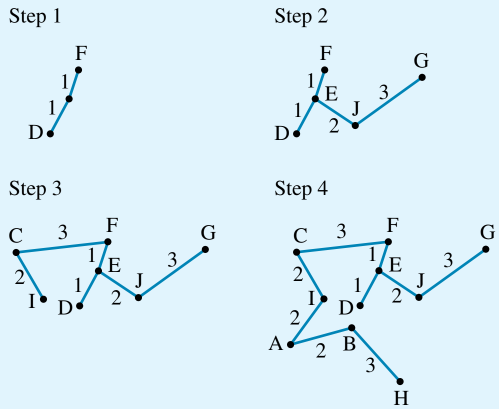
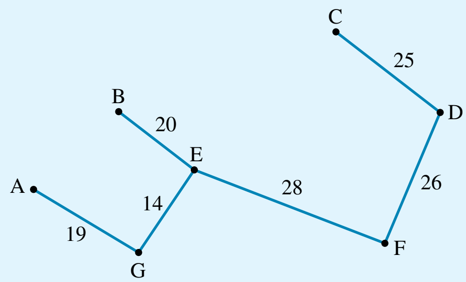
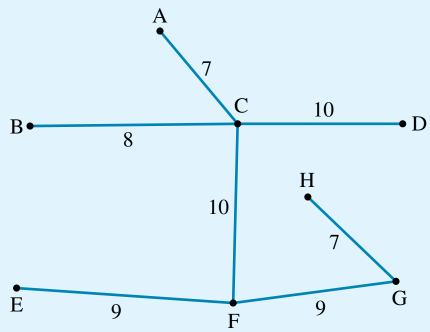


10 a



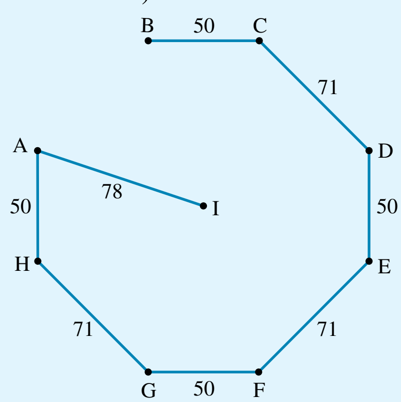
b



11 a Longest: IFEDCBAHG (or similar variation of the same values)

Shortest: IAHGFEDCB (or similar variation of the same values)

b



12 a FDCGBAE (other solutions exist)

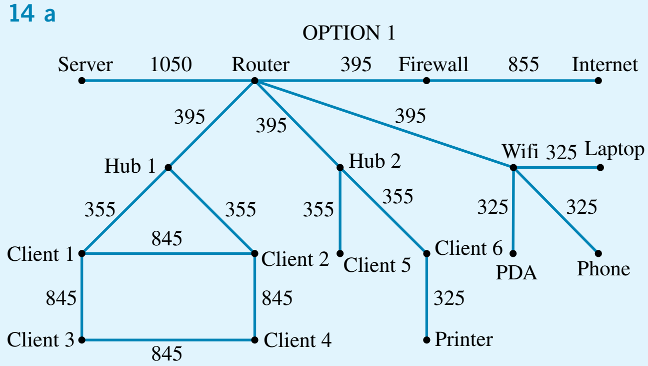
b FDCBAEG (other solutions exist)

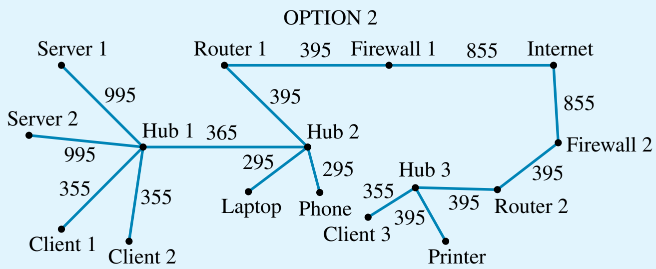
13 a ADEG

b BHG

c EGFCDABHE

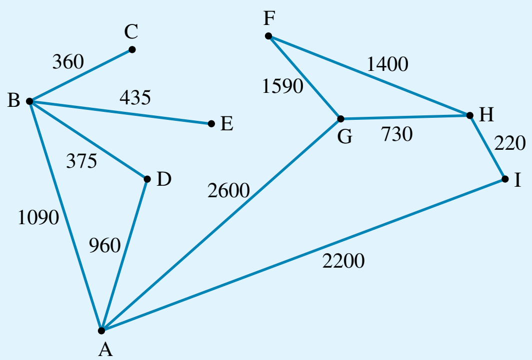
14 a





b Option 2

15 a

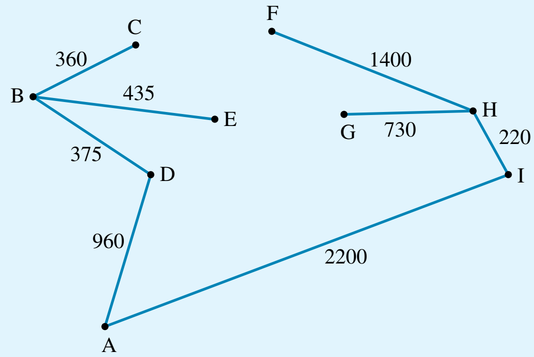


b No; C and E are both only reachable from B.

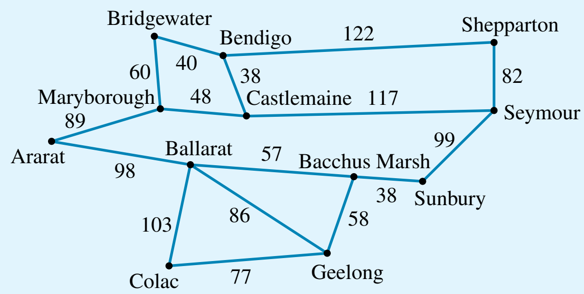
c **i** 12 025

ii 12 025

d

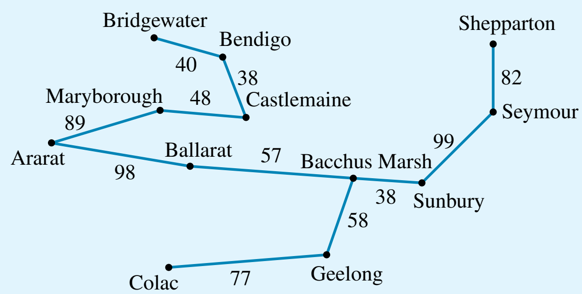


16 a



b 723 km

c



d 859 km