4

Matrices

4.1 Kick off with CAS

Using CAS to work with matrices

1 **a** Define each of the following matrices using CAS.

$$A = \begin{bmatrix} -2 & 3 \\ -2 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b Using the matrices defined in part **a**, calculate:

i $5A$ **ii** $2B$ **iii** $2A + 3B$

iv $\det A$ **v** B^{-1} **vi** BI.

2 **a** Define the following matrices using CAS.

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

b Using the matrices defined in part **a**, find X if:

i $AX = B$ **ii** $XA = B$.

3 Solve the simultaneous equations

$$9x + 10y = 153$$
$$3x - y = 12$$

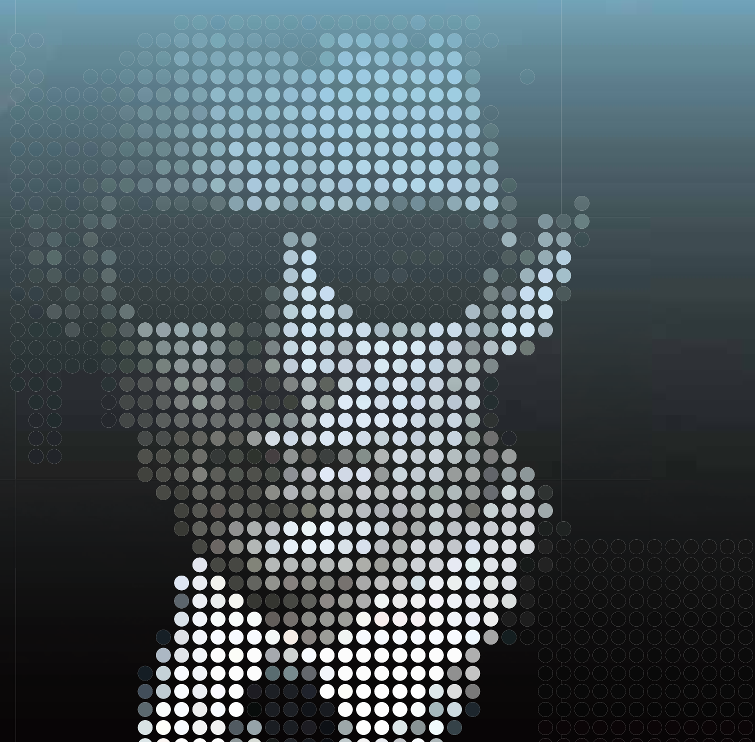
by setting up the simultaneous equations in the form

$$\begin{bmatrix} 9 & 10 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 153 \\ 12 \end{bmatrix}$$

and completing the following steps.

a Define $A = \begin{bmatrix} 9 & 10 \\ 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 153 \\ 12 \end{bmatrix}$ using CAS.

b Using the matrices defined in part **a**, solve the equation $AX = B$ for X.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

4.2 Types of matrices

Matrices

studyon

Units 1 & 2
AOS 3
Topic 1
Concept 1

Definition of a matrix
Concept summary
Practice questions

A **matrix** is a rectangular array of rows and columns that is used to store and display information. Matrices can be used to represent many different types of information, such as the models of cars sold in different car dealerships, the migration of people to different countries and the shopping habits of customers at different department stores. Matrices also play an important role in encryption. Before sending important information, programmers encrypt or code messages using matrices; the people receiving the information will then use inverse matrices as the key to decode the message. Engineers, scientists and project managers also use matrices to help them to perform various everyday tasks.

Describing matrices

A matrix is usually displayed in square brackets with no borders between the rows and columns.

The table below left shows the number of participants attending three different dance classes (rumba, waltz and chacha) over the two days of a weekend. The matrix below right displays the information presented in the table.

Number of participants attending the dance classes

	Saturday	Sunday
Rumba	9	13
Tango	12	8
Chacha	16	14

Matrix displaying the number of participants attending the dance classes

$$\begin{bmatrix} 9 & 13 \\ 12 & 8 \\ 16 & 14 \end{bmatrix}$$

WORKED EXAMPLE 1

The table below shows the number of adults and children who attended three different events over the school holidays. Construct a matrix to represent this information.

	Circus	Zoo	Show
Adults	140	58	85
Children	200	125	150

THINK	WRITE
1 A matrix is like a table that stores information. What information needs to be displayed?	The information to be displayed is the number of adults and children attending the three events: circus, zoo and show.

2 Write down how many adults and children attend each of the three events

	Circus	Zoo	Show
Adults	140	58	85
Children	200	125	150

3 Write this information in a matrix. Remember to use square brackets.

$$\begin{bmatrix} 140 & 58 & 85 \\ 200 & 125 & 150 \end{bmatrix}$$

Networks

Matrices can also be used to display information about various types of **networks**, including road systems and social networks. The following matrix shows the links between a group of schoolmates on Facebook, with a 1 indicating that the two people are friends on Facebook and a 0 indicating that the two people aren't friends on Facebook.

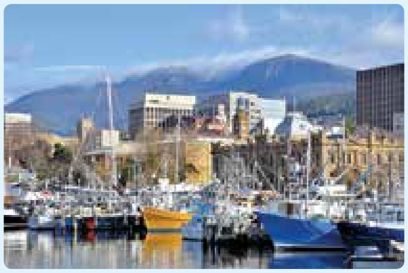
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

From this matrix you can see that the following people are friends with each other on Facebook:

- person A and person B
- person A and person C
- person B and person D
- person C and person D.

WORKED EXAMPLE 2

The distances, in kilometres, along three major roads between the Tasmanian towns Launceston (L), Hobart (H) and Devonport (D) are displayed in the matrix below.



$$\begin{array}{c} \\ H \\ D \\ L \end{array} \begin{array}{c} \begin{matrix} H & D & L \end{matrix} \\ \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$$

a **What is the distance, in kilometres, between Devonport and Hobart?**

b **Victor drove 75 km directly between two of the Tasmanian towns. Which two towns did he drive between?**

c **The Goldstein family would like to drive from Hobart to Launceston, and then to Devonport. Determine the total distance in kilometres that they will travel.**

THINK	WRITE
a 1 Reading the matrix, locate the first city or town, i.e. Devonport (D), on the top of the matrix	**a** $$\begin{array}{cc} & \begin{array}{ccc} H & D & L \end{array} \\ \begin{array}{c} H \\ D \\ L \end{array} & \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$$
2 Locate the second city or town, i.e. Hobart (H), on the side of the matrix.	$$\begin{array}{cc} & \begin{array}{ccc} H & D & L \end{array} \\ \begin{array}{c} \rightarrow H \\ D \\ L \end{array} & \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$$
3 The point where both arrows meet gives you the distance between the two towns.	207 km
b 1 Locate the entry '75' in the matrix.	**b** $$\begin{array}{cc} & \begin{array}{ccc} H & D & L \end{array} \\ \begin{array}{c} H \\ D \\ L \end{array} & \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & \textcircled{75} \\ 160 & 75 & 0 \end{bmatrix} \end{array}$$
2 Locate the column and row 'titles' (L and D) for that entry.	$$\begin{array}{cc} & \begin{array}{ccc} H & D & L \end{array} \\ \begin{array}{c} H \\ D \\ L \end{array} & \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & \textcircled{75} \\ 160 & 75 & 0 \end{bmatrix} \end{array}$$
3 Refer to the title headings in the question.	Victor drove between Launceston and Devonport.
c 1 Locate the first city or town, i.e. Hobart (H), on the top of the matrix and the second city or town, i.e. Launceston (L), on the side of the matrix.	**c** $$\begin{array}{cc} & \begin{array}{ccc} H & D & L \end{array} \\ \begin{array}{c} H \\ D \\ \rightarrow L \end{array} & \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$$
2 Where the row and column meet gives the distance between the two towns.	160 km
3 Determine the distance between the second city or town, i.e. Launceston, and the third city or town, i.e. Devonport.	$$\begin{array}{cc} & \begin{array}{ccc} H & D & L \end{array} \\ \begin{array}{c} H \\ \rightarrow D \\ L \end{array} & \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$$
4 Where the row and column meet gives the distance between the two towns.	75 km
5 Add the two distances together.	$160 + 75 = 235$ km

Defining matrices

The **order** of a matrix is defined by the number of rows, m, and number of columns, n, in the matrix.

Consider the following matrix, A.

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & -6 & 5 \end{bmatrix}$$

Matrix A has two rows and three columns, and its order is 2×3 (read as a 'two by three' matrix).

A matrix that has the same number of rows and columns is called a **square matrix**.

$$B = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$$

Matrix B has two rows and two columns and is a 2×2 square matrix.

A **row matrix** has only one row.

$$C = \begin{bmatrix} 3 & 7 & -4 \end{bmatrix}$$

Matrix C has only one row and is called a row matrix.

A **column matrix** has only one column.

$$D = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

Matrix D has only one column and is called a column matrix.

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Units 1 & 2
AOS 3
Topic 1
Concept 2

Naming and uses of matrices
Concept summary
Practice questions

WORKED EXAMPLE 3

At High Vale College, 150 students are studying General Mathematics and 85 students are studying Mathematical Methods. Construct a column matrix to represent the number of students studying General Mathematics and Mathematical Methods, and state the order of the matrix.

THINK	WRITE
1 Read the question and highlight the key information	150 students study General Mathematics. 85 students study Mathematical Methods.
2 Display this information in a column matrix.	$\begin{bmatrix} 150 \\ 85 \end{bmatrix}$
3 How many rows and columns are there in this matrix?	The order of the matrix is 2×1.

Elements of matrices

The entries in a matrix are called **elements**. The position of an element is described by the corresponding row and column. For example, a_{21} means the entry in the 2nd row and 1st column of matrix A, as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

WORKED EXAMPLE 4

Write the element a_{23} for the matrix $A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & -6 & 5 \end{bmatrix}$.

THINK	WRITE
1 The element a_{23} means the element in the 2nd row and 3rd column. Draw lines through the 2nd row and 3rd column to help you identify this element.	$A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & -6 & 5 \end{bmatrix}$
2 Identify the number that is where the lines cross over.	$a_{23} = 5$

Identity matrices

An **identity matrix**, *I*, is a square matrix in which all of the elements on the diagonal line from the top left to bottom right are 1s and all of the other elements are 0s.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are both identity matrices.

As you will see later in this topic, identity matrices are used to find inverse matrices, which help solve matrix equations.

The zero matrix

A **zero matrix**, *0*, is a square matrix that consists entirely of '0' elements.

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an example of a zero matrix.

EXERCISE 4.2 Types of matrices

PRACTISE

1 WE1 Cheap Auto sells three types of vehicles: cars, vans and motorbikes. They have two outlets at Valley Heights and Hill Vale. The number of vehicles in stock at each of the two outlets is shown in the table.

	Cars	Vans	Motorbikes
Valley Heights	18	12	8
Hill Vale	13	10	11

Construct a matrix to represent this information.

2 Newton and Isaacs played a match of tennis. Newton won the match in five sets with a final score of 6–2, 4–6, 7–6, 3–6, 6–4. Construct a matrix to represent this information.

3 WE2 The distance in kilometres between the towns Port Augusta (P), Coober Pedy (C) and Alice Springs (A) are displayed in the following matrix.

$$\begin{array}{c} \\ P \\ C \\ A \end{array}\begin{array}{c} \begin{array}{ccc} P & C & A \end{array} \\ \begin{bmatrix} 0 & 545 & 1225 \\ 545 & 0 & 688 \\ 1225 & 688 & 0 \end{bmatrix} \end{array}$$

a Determine the distance in kilometres between Port Augusta and Coober Pedy.

b Greg drove 688 km between two towns. Which two towns did he travel between?

c A truck driver travels from Port Augusta to Coober Pedy, then onto Alice Springs. He then drives from Alice Springs directly to Port Augusta. Determine the total distance in kilometres that the truck driver travelled.

4 A one-way economy train fare between Melbourne Southern Cross Station and Canberra Kingston Station is $91.13. A one-way economy train fare between Sydney Central Station and Melbourne Southern Cross Station is $110.72, and a one-way economy train fare between Sydney Central Station and Canberra Kingston Station is $48.02.

a Represent this information in a matrix.

b Drew travelled from Sydney Central to Canberra Kingston Station, and then onto Melbourne Southern Cross. Determine how much, in dollars, he paid for the train fare.

5 WE3 An energy-saving store stocks shower water savers and energy-saving light globes. In one month they sold 45 shower water savers and 30 energy-saving light globes. Construct a column matrix to represent the number of shower water savers and energy-saving light globes sold during this month, and state the order of the matrix.

6 Happy Greens Golf Club held a three-day competition from Friday to Sunday. Participants were grouped into three different categories: experienced, beginner and club member. The table shows the total entries for each type of participant on each of the days of the competition.

Category	Friday	Saturday	Sunday
Experienced	19	23	30
Beginner	12	17	18
Club member	25	33	36

a How many entries were received for the competition on Friday?

b Calculate the total number of entries for the three day competition.

c Construct a row matrix to represent the number of beginners participating in the competition for each of the three days.

7 WE4 Write down the value of the following elements for matrix D.

$$D = \begin{bmatrix} 4 & 5 & 0 \\ 2 & -1 & -3 \\ 1 & -2 & 6 \\ 0 & 3 & 7 \end{bmatrix}$$

a d_{12} b d_{33} c d_{43}

8 Consider the matrix $E = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{4} \\ -1 & -\frac{1}{2} & -3 \end{bmatrix}$.

a Explain why the element e_{24} does not exist.
b Which element has a value of -3?
c Nadia was asked to write down the value of element e_{12} and wrote -1. Explain Nadia's mistake and state the correct value of element e_{12}.

CONSOLIDATE

9 Write the order of matrices A, B and C.

$$A = [3], \quad B = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad C = [4 \quad -2]$$

10 Which of the following represent matrices? Justify your answers.

a $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

b $\begin{bmatrix} 4 & 0 \\ & 3 \end{bmatrix}$

c $\begin{bmatrix} & 5 \\ 4 & \\ & 7 \end{bmatrix}$

d $\begin{bmatrix} a & c & e & g \\ b & d & f & h \end{bmatrix}$

11 Matrices D and E are shown. Write down the value of the following elements.

$$D = \begin{bmatrix} 5 & 0 & 2 & -1 \\ 8 & 1 & 3 & 6 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.5 & 0.3 \\ 1.2 & 1.1 \\ 0.4 & 0.9 \end{bmatrix}$$

a d_{23}
b d_{14}
c d_{22}
d e_{11}
e e_{32}

12 a The following matrix represents an incomplete 3×3 identity matrix. Complete the matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & & 0 \\ & 0 & \end{bmatrix}$$

b Construct a 2×2 zero matrix.

13 The elements in matrix H are shown below.

$h_{12} = 3$

$h_{11} = 4$

$h_{21} = -1$

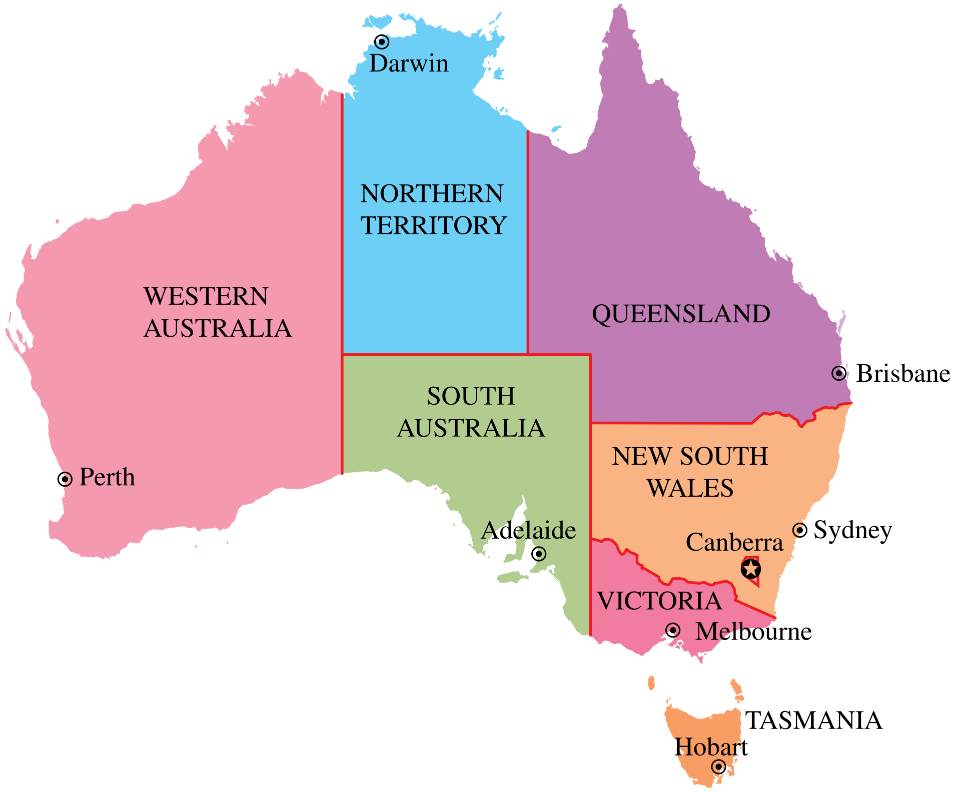
$h_{31} = -4$

$h_{32} = 6$

$h_{22} = 7$

a State the order of matrix H.
b Construct matrix H.

14 The land area and population of each Australian state and territory were recorded and summarised in the table below.



State/territory	Land area (km^2)	Population (millions)
Australian Capital Territory	2358	0.4
Queensland	1 727 200	4.2
New South Wales	801 428	6.8
Northern Territory	1 346 200	0.2
South Australia	984 000	1.6
Western Australia	2 529 875	2.1
Tasmanian	68 330	0.5
Victoria	227 600	5.2

a Construct an 8×1 matrix that displays the population, in millions, of each state and territory in the order shown in the table.

b Construct a row matrix that represents the land area of each of the states in ascending order.

c Town planners place the information on land area, in km^2, and population, in millions, for the states New South Wales, Victoria and Queensland respectively in a matrix.

  i State the order of this matrix.

  ii Construct this matrix.

15 The estimated number of Indigenous Australians living in each state and territory in Australia in 2006 is shown in the following table.

State and territory	Number of Indigenous persons	% of population that is Indigenous
New South Wales	148 178	2.2
Victoria	30 839	0.6
Queensland	146 429	3.6
South Australia	26 044	1.7
Western Australia	77 928	3.8
Tasmania	16 900	3.4
Northern Territory	66 582	31.6
Australian Capital Territory	4 043	1.2

a Construct an 8×2 matrix to represent this information

b Determine the total number of Indigenous persons living in the following states and territories in 2006:

i Northern Territory

ii Tasmania

iii Queensland, New South Wales and Victoria (combined).

c Determine the total number of Indigenous persons who were estimated to be living in Australia in 2006.

16 AeroWings is a budget airline specialising in flights between four mining towns: Olympic Dam (O), Broken Hill (B), Dampier (D) and Mount Isa (M).

The cost of airfares (in dollars) to fly from the towns in the top row to the towns in the first column is shown in the matrix below.

$$\text{To}\ \begin{matrix} \\ O \\ B \\ D \\ M \end{matrix} \overset{\text{From}}{\begin{matrix} \begin{matrix} O & B & D & M \end{matrix} \\ \begin{bmatrix} 0 & 70 & 150 & 190 \\ 89 & 0 & 85 & 75 \\ 175 & 205 & 0 & 285 \\ 307 & 90 & 101 & 0 \end{bmatrix} \end{matrix}}$$

a In the context of this problem, explain the meaning of the zero entries.

b Find the cost, in dollars, to fly from Olympic Dam to Dampier.

c Yen paid \$101 for his airfare with AeroWings. At which town did he arrive?

d AeroWings offers a 25% discount for passengers flying between Dampier and Mount Isa, and a 15% discount for passengers flying from Broken Hill to Olympic Dam. Construct another matrix that includes the discounted airfares (in dollars) between the four mining towns.

17 The matrix below displays the number of roads connecting five towns: Ross (R), Stanley (S), Thomastown (T), Edenhope (E) and Fairhaven (F).

$$N = \begin{array}{c} \\ \end{array}\begin{matrix} & \text{R} & \text{S} & \text{T} & \text{E} & \text{F} \end{matrix}$$

$$N = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \text{R} \\ \text{S} \\ \text{T} \\ \text{E} \\ \text{F} \end{matrix}$$

a Construct a road map using the information shown.

b Determine whether the following statements are true or false.

i There is a road loop at Stanley.

ii You can travel directly between Edenhope and Stanley.

iii There are two roads connecting Thomastown and Edenhope.

iv There are only three different ways to travel between Ross and Fairhaven.

c A major flood washes away part of the road connecting Ross and Thomastown. Which elements in matrix N will need to be changed to reflect the new road conditions between the towns?

18 Mackenzie is sitting a Mathematics multiple choice test with ten questions. There are five possible responses for each question: A, B, C, D and E. She selects A for the first question and then determines the answers to the remaining questions using the following matrix.

$$\begin{matrix} & & \text{This question} \\ & & \begin{matrix} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{matrix} \end{matrix}$$

$$\text{Next question} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

a Using the matrix above, what is Mackenzie's answer to question 2 on the test?

b Write Mackenzie's responses to the remaining eight questions.

c Explain why it is impossible for Mackenzie to have more than one answer with response A.

Mackenzie used another matrix to help her answer the multiple choice test. Her responses using this matrix are shown in this grid.

Question	1	2	3	4	5	6	7	8	9	10
Response	A	D	C	B	E	A	D	C	B	E

d Complete the matrix that Mackenzie used for the test by finding the values of the missing elements.

$$\begin{array}{cc} & \text{This question} \\ & \begin{array}{ccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{array} \\ \text{Next question} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & \\ 0 & & & 0 & 0 \\ 0 & & & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & & 0 & 0 & \end{bmatrix} \end{array}$$

MASTER

19 State the steps involved in constructing a matrix using CAS.

20 Matrix A was constructed using a spreadsheet.

$$A = \begin{bmatrix} 75 & 80 & 55 \\ 120 & 65 & 82 \\ 95 & 105 & 71 \end{bmatrix}$$

a State the cell number for each of the following elements.

i a_{13} ii a_{22}

iii a_{32} iv a_{21}

b i Explain how the element position can be used to locate the corresponding cell number in the spreadsheet.

ii Using your response to **bi**, write down the cell number for any element e_{nm}.

4.3 Operations with matrices

Matrix addition and subtraction

eBook*plus*

Interactivity
Adding and subtracting matrices
int-6463

Matrices can be added and subtracted using the same rules as in regular arithmetic. However, matrices can only be added and subtracted if they are the same order (that is, if they have the same number of rows and columns).

Adding matrices

To add matrices, you need to add the corresponding elements of each matrix together (that is, the numbers in the same position).

WORKED EXAMPLE 5

If $A = \begin{bmatrix} 4 & 2 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix}$, find the value of $A + B$.

THINK	WRITE
1 Write down the two matrices in a sum.	$\begin{bmatrix} 4 & 2 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix}$
2 Identify the elements in the same position. For example, 4 and 1 are both in the first row and first column. Add the elements in the same positions together.	$\begin{bmatrix} 4+1 & 2+0 \\ 3+5 & -2+3 \end{bmatrix}$
3 Work out the sums and write the answer.	$\begin{bmatrix} 5 & 2 \\ 8 & 1 \end{bmatrix}$

Subtracting matrices

To subtract matrices, you need to subtract the corresponding elements in the same order as presented in the question.

WORKED EXAMPLE 6 If $A = \begin{bmatrix} 6 & 0 \\ 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, find the value of $A - B$.

THINK	WRITE
1 Write the two matrices.	$\begin{bmatrix} 6 & 0 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$
2 Subtract the elements in the same position together.	$\begin{bmatrix} 6-4 & 0-2 \\ 2-1 & -2-3 \end{bmatrix}$
3 Work out the subtractions and write the answer.	$\begin{bmatrix} 2 & -2 \\ 1 & -5 \end{bmatrix}$

EXERCISE 4.3 Operations with matrices

PRACTISE

studyon

Units 1 & 2
AOS 3
Topic 1
Concept 3

Equality, addition and subtraction
Concept summary
Practice questions

1 a WE5 If $A = \begin{bmatrix} 2 & -3 \\ -1 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 9 \\ 0 & 11 \end{bmatrix}$, find the value of $A + B$.

b If $A = \begin{bmatrix} 0.5 \\ 0.1 \\ 1.2 \end{bmatrix}$, $B = \begin{bmatrix} -0.5 \\ 2.2 \\ 0.9 \end{bmatrix}$ and $C = \begin{bmatrix} -0.1 \\ -0.8 \\ 2.1 \end{bmatrix}$, find the matrix sum $A + B + C$.

2 Consider the matrices $C = \begin{bmatrix} 1 & -3 \\ 7 & 5 \\ b & 8 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & a \\ -5 & -4 \\ 2 & -9 \end{bmatrix}$.

If $C + D = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -4 & -1 \end{bmatrix}$, find the values of a and b.

3 WE6 If $A = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -2 & 4 \end{bmatrix}$, find the value of $A - B$.

4 Consider the following.

$$B - A = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, A + B = \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

a Explain why matrix B must have an order of 3×1.

b Determine matrix B.

CONSOLIDATE

5 If $A = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$ and $C = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$, calculate the following.

a $A + C$ b $B + C$ c $A - B$ d $A + B - C$

6 Evaluate the following.

a $[0.5 \quad 0.25 \quad 1.2] - [0.75 \quad 1.2 \quad 0.9]$

b $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 6 & 0 \end{bmatrix}$

c $\begin{bmatrix} 12 & 17 & 10 \\ 35 & 20 & 25 \\ 28 & 32 & 29 \end{bmatrix} - \begin{bmatrix} 13 & 12 & 9 \\ 31 & 22 & 22 \\ 25 & 35 & 31 \end{bmatrix}$

d $\begin{bmatrix} 11 & 6 & 9 \\ 7 & 12 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 8 & 8 \\ 6 & 7 & 6 \end{bmatrix} - \begin{bmatrix} -2 & -1 & 10 \\ 4 & 9 & -3 \end{bmatrix}$

7 If $\begin{bmatrix} 3 & 0 \\ 5 & a \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ -b & 1 \end{bmatrix} = \begin{bmatrix} c & 2 \\ 3 & -4 \end{bmatrix}$, find the values of a, b and c.

8 If $\begin{bmatrix} 12 & 10 \\ 25 & 13 \\ 20 & a \end{bmatrix} - \begin{bmatrix} 9 & 11 \\ 26 & c \\ b & 9 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 8 \\ 21 & -3 \end{bmatrix}$, find the values of a, b and c.

9 By finding the order of each of the following matrices, identify which of the matrices can be added and/or subtracted to each other and explain why.

$A = [1 \quad -5]$ $\qquad B = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$ $\qquad C = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$

$D = [-4]$ $\qquad E = [-3 \quad 6]$

10 Hard Eggs sells both free-range and barn-laid eggs in three different egg sizes (small, medium and large) to two shops, Appleton and Barntown. The number of cartons ordered for the Appleton shop is shown in the table below.

Eggs	Small	Medium	Large
Free range	2	3	5
Barn laid	4	6	3

a Construct a 2×3 matrix to represent the egg order for the Appleton shop.
The total orders for both shops are shown in the table below.

Eggs	Small	Medium	Large
Free range	3	4	8
Barn laid	6	8	5

b i Set up a matrix sum that would determine the order for the Barntown shop.
ii Use the matrix sum from part bi to determine the order for the Barntown shop. Show the order in a table.

11 Marco was asked to complete the matrix sum $\begin{bmatrix} 8 & 126 & 59 \\ 17 & 102 & -13 \end{bmatrix} + \begin{bmatrix} 22 & 18 & 38 \\ 16 & 27 & 45 \end{bmatrix}$.
He gave $\begin{bmatrix} 271 \\ 194 \end{bmatrix}$ as his answer.

a By referring to the order of matrices, explain why Marco's answer must be incorrect.

b By explaining how to add matrices, write simple steps for Marco to follow so that he is able to add and subtract any matrices. Use the terms 'order of matrices' and 'elements' in your explanation.

12 Frederick, Harold, Mia and Petra are machinists who work for Stitch in Time. The table below shows the hours worked by each of the four employees and the number of garments completed each week for the last three weeks.

Employee	Week 1		Week 2		Week 3	
	Hours worked	Number of garments	Hours worked	Number of garments	Hours worked	Number of garments
Frederick	35	150	32	145	38	166
Harold	41	165	36	152	35	155
Mia	38	155	35	135	35	156
Petra	25	80	30	95	32	110

a Construct a 4 × 1 matrix to represent the number of garments each employee made in week 1.

b **i** Create a matrix sum that would determine the total number of garments each employee made over the three weeks.

ii Using your matrix sum from part **bi**, determine the total number of garments each employee made over the three weeks.

c Nula is the manager of Stitch in Time. She uses the following matrix sum to determine the total number of hours worked by each of the four employees over the three weeks.

$$\begin{bmatrix} 35 \\ \\ 38 \\ 25 \end{bmatrix} + \begin{bmatrix} \\ 36 \\ \\ 30 \end{bmatrix} + \begin{bmatrix} 38 \\ 35 \\ 35 \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Complete the matrix sum by filling in the missing values.

13 There are three types of fish in a pond: speckles, googly eyes and fantails. At the beginning of the month there were 12 speckles, 9 googly eyes and 8 fantails in the pond. By the end of the month there were 9 speckles, 6 googly eyes and 8 fantails in the pond.

a Construct a matrix sum to represent this information.

b After six months, there were 12 speckles, 4 googly eyes and 10 fantails in the pond. Starting from the end of the first month, construct another matrix sum to represent this information.

14 Consider the following matrix sum: $A - C + B = D$. Matrix D has an order of 3×2.

a State the order of matrices A, B and C. Justify your answer.

A has elements $a_{11} = x$, $a_{21} = 20$, $a_{31} = 3c_{31}$, $a_{12} = 7$, $a_{22} = y$ and $a_{32} = -8$.

B has elements $b_{11} = x$, $b_{21} = 2x$, $b_{31} = 3x$, $b_{12} = y$, $b_{22} = 5$ and $b_{32} = 6$

C has elements $c_{11} = 12$, $c_{21} = \frac{1}{2}a_{21}$, $c_{31} = 5$, $c_{12} = 9$, $c_{22} = 2y$ and $c_{32} = 2x$.

b Define the elements of D in terms of x and y.

c If $D = \begin{bmatrix} -8 & 1 \\ 14 & 2 \\ 16 & -8 \end{bmatrix}$, show that $x = 2$ and $y = 3$.

MASTER

15 Using CAS or otherwise, evaluate the matrix sum

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{3}{5} & \frac{2}{7} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{4} & \frac{2}{9} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{1}{3} \\ \frac{1}{10} & \frac{3}{14} & \frac{4}{9} \\ \frac{1}{6} & \frac{1}{2} & \frac{2}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{7}{6} \\ \frac{2}{15} & \frac{8}{21} & \frac{4}{3} \\ \frac{2}{9} & \frac{5}{8} & \frac{10}{9} \end{bmatrix}.$$

16 Consider the matrices A and B.

$$A = \begin{bmatrix} 21 & 10 & 9 \\ 18 & 7 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} -10 & 19 & 11 \\ 36 & -2 & 15 \end{bmatrix}$$

The matrix sum $A + B$ was performed using a spreadsheet. The elements for A were entered into a spreadsheet in the following cells: a_{11} was entered in cell A1, a_{21} into cell A2, a_{12} in cell B1, a_{22} in cell B2, a_{13} in cell C2 and a_{23} in cell C3.

a If the respective elements for B were entered into cells D1, D2, E1, E2, F1 and F2, write the formulas required to find the matrix sum $A + B$.

b Hence, using a spreadsheet, state the elements of $A + B$.

4.4 Matrix multiplication

Scalar multiplication

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If $A = \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 0 & 7 \end{bmatrix}$, then $A + A$ can be found by multiplying each element in matrix A by the scalar number 2, because $A + A = 2A$.

$$A + A = \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 10 & 2 \\ 0 & 14 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 \times 3 & 2 \times 2 \\ 2 \times 5 & 2 \times 1 \\ 2 \times 0 & 2 \times 7 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 10 & 2 \\ 0 & 14 \end{bmatrix}$$

The number 2 is known as a scalar quantity, and the matrix $2A$ represents a **scalar multiplication**. Any matrix can be multiplied by any scalar quantity and the order of the matrix will remain the same. A scalar quantity can be any real number, such as negative or positive numbers, fractions or decimals numbers.

WORKED EXAMPLE 7

Consider the matrix $A = \begin{bmatrix} 120 & 90 \\ 80 & 60 \end{bmatrix}$.

Evaluate the following.

a $\frac{1}{4}A$

b $0.1A$

THINK / **WRITE**

a 1 Identify the scalar for the matrix. In this case it is $\frac{1}{4}$, which means that each element in A is multiplied by $\frac{1}{4}$ (or divided by 4).

$\frac{1}{4}\begin{bmatrix} 120 & 90 \\ 80 & 60 \end{bmatrix}$

2 Multiply each element in A by the scalar.

$\begin{bmatrix} \frac{1}{4} \times 120 & \frac{1}{4} \times 90 \\ \frac{1}{4} \times 80 & \frac{1}{4} \times 60 \end{bmatrix}$

3 Simplify each multiplication by finding common factors and write the answer.

$\begin{bmatrix} \cancel{120}^{30} \times \frac{1}{\cancel{4}^{1}} & \cancel{90}^{45} \times \frac{1}{\cancel{4}^{2}} \\ \cancel{80}^{20} \times \frac{1}{\cancel{4}^{1}} & \cancel{60}^{15} \times \frac{1}{\cancel{4}^{1}} \end{bmatrix}$

$= \begin{bmatrix} 30 & \frac{45}{2} \\ 20 & 15 \end{bmatrix}$

b 1 Identify the scalar. In this case it is 0.1, which means that each element in A is multiplied by 0.1 (or divided by 10).

$0.1\begin{bmatrix} 120 & 90 \\ 80 & 60 \end{bmatrix}$

2 Multiply each element in A by the scalar.

$\begin{bmatrix} 0.1 \times 120 & 0.1 \times 90 \\ 0.1 \times 80 & 0.1 \times 60 \end{bmatrix}$

3 Find the values for each element and write the answer.

$\begin{bmatrix} 12 & 9 \\ 8 & 6 \end{bmatrix}$

The product matrix and its order

Not all matrices can be multiplied together. However, unlike with addition and subtraction, matrices do not need to have the same order to be multiplied together.

For matrices to be able to be multiplied together (have a product), the *number of columns* in the first matrix must equal the *number of rows* in the second matrix.

For example, consider matrices A and B, with matrix A having an order of $m \times n$ (m rows and n columns) and matrix B having an order of $p \times r$ (p rows and r columns).

For A and B to be multiplied together, the number of columns in A must equal the number of rows in B; that is, n must equal p. If n does equal p, then the **product matrix** AB is said to exist, and the order of the product matrix AB will be $m \times r$.

Given matrix A with an order of $m \times n$ and matrix B with an order of $n \times r$, matrix AB will have an order of $m \times r$.

WORKED EXAMPLE 8

If $A = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $B = [1 \quad 2]$, show that the product matrix AB exists and hence write down the order of AB.

THINK	WRITE
1 Write the order of each matrix.	A: 2×1 B: 1×2
2 Write the orders next to each other.	$2 \times 1 \; 1 \times 2$
3 Circle the two middle numbers.	$2 \times (1 \; 1) \times 2$
4 If the two numbers are the same, then the product matrix exists.	Number of columns in A = number of rows in B, therefore the product matrix AB exists.
5 The order of the resultant matrix (the product) will be the first and last number.	$(2) \times 1 \; 1 \times (2)$ The order of AB is 2×2.

Multiplying matrices

To multiply matrices together, use the following steps.

Step 1: Confirm that the product matrix exists (that is, the number of columns in the first matrix equals the number of rows in the second matrix).

Step 2: Multiply the elements of each row of the first matrix by the elements of each column of the second matrix.

Step 3: Sum the products in each element of the product matrix.

Consider matrices A and B.

$$A = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } B = [1 \quad 2]$$

As previously stated, the order of the product matrix AB will be 2×2.

$$AB = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \times [1 \quad 2]$$

1st row × 1st column: 3×1

1st row × 2nd column: 3×2

2nd row × 1st column: 2×1

2nd row × 2nd column: 2×2

$$AB = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

Unlike when multiplying with real numbers, when multiplying matrices together the order of the multiplication is important. This means that in most cases $AB \neq BA$.

Using matrices A and B as previously defined, the order of product matrix BA is 1×1.

$$BA = [1 \quad 2] \times \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

As when calculating AB, to multiply the elements in these matrices you need to multiply the rows by the columns. Each element in the first row must be multiplied by the corresponding element in the first column, and the total sum of these will make up the element in the first row and first column of the product matrix.

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For example, the element in the first row and first column of the product matrix BA is found by the sum $1 \times 3 + 2 \times 2 = 7$.

So the product matrix BA is $[7]$.

WORKED EXAMPLE 9 **If $A = [3 \quad 5]$ and $B = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, determine the product matrix AB.**

THINK / **WRITE**

1 Set up the product matrix.

$[3 \quad 5] \times \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

2 Determine the order of product matrix AB by writing the order of each matrix A and B

$A \times B$

$\textcircled{1} \times 2 \times 2 \times \textcircled{1}$

AB has an order of 1×1.

3 Multiply each element in the first row by the corresponding element in the first column; then calculate the sum of the results.

$3 \times 2 + 5 \times 6 = 36$

4 Write the answer as a matrix.

$[36]$

WORKED EXAMPLE 10 **Determine the product matrix MN if M = $\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix}$ and N = $\begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$.**

THINK / **WRITE**

1 Set up the product matrix

$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$

2 Determine the order of product matrix MN by writing the order of each matrix M and N

$M \times N$

$\textcircled{2} \times 2 \times 2 \times \textcircled{2}$

MN has an order of 2×2.

3 To find the element MN_{11}, multiply the corresponding elements in the first row and first column and calculate the sum of the results.

$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$

$3 \times 1 + 6 \times 5 = 33$

4 To find the element MN_{12}, multiply the corresponding elements in the first row and second column and calculate the sum of the results

$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$

$3 \times 8 + 6 \times 4 = 48$

5 To find the element MN_{21}, multiply the corresponding elements in the second row and first column and calculate the sum of the results.

$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$

$5 \times 1 + 2 \times 5 = 15$

6 To find the element MN_{22}, multiply the corresponding elements in the second row and second column and calculate the sum of the results.

$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$

$5 \times 8 + 2 \times 4 = 48$

7 Construct the matrix MN by writing in each of the elements.

$\begin{bmatrix} 33 & 48 \\ 15 & 48 \end{bmatrix}$

Multiplying by the identity matrix

As previously stated, an identity matrix is a square matrix with 1s in the top left to bottom right diagonal and 0s for all other elements,

for example $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Just like multiplying by the number 1 in the real number system, multiplying by the identity matrix will not change a matrix.

If the matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$ is multiplied by the identity matrix on the left, that is IA, it will be multiplied by a 2×2 identity matrix (because A has 2 rows). If A is multiplied by the identity matrix on the right, that is AI, then it will be multiplied by a 3×3 identity matrix (because A has 3 columns).

$$\begin{aligned} IA &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 2 + 0 \times 3 & 1 \times 4 + 0 \times 5 & 1 \times 6 + 0 \times 7 \\ 0 \times 2 + 1 \times 3 & 0 \times 4 + 1 \times 5 & 0 \times 6 + 1 \times 7 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \\ &= A \end{aligned}$$

$$\begin{aligned} AI &= \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 1 + 4 \times 0 + 6 \times 0 & 2 \times 0 + 4 \times 1 + 6 \times 0 & 2 \times 0 + 4 \times 0 + 6 \times 1 \\ 3 \times 1 + 5 \times 0 + 7 \times 0 & 3 \times 0 + 5 \times 1 + 7 \times 0 & 3 \times 0 + 5 \times 0 + 7 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \\ &= A \end{aligned}$$

Therefore, $AI = IA = A$.

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Powers of square matrices

When a square matrix is multiplied by itself, the order of the resultant matrix is equal to the order of the original square matrix. Because of this fact, whole number powers of square matrices always exist.

You can use CAS to quickly determine large powers of square matrices.

WORKED EXAMPLE 11

If $A = \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}$, calculate the value of A^3.

THINK	WRITE
1 Write the matrix multiplication in full.	$A^3 = AAA$ $= \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}$

2 Calculate the first matrix multiplication (AA).

$$AA = \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}$$

$$AA_{11} = 3 \times 3 + 5 \times 5 = 34$$
$$AA_{21} = 5 \times 3 + 1 \times 5 = 20$$
$$AA_{12} = 3 \times 5 + 5 \times 1 = 20$$
$$AA_{22} = 5 \times 5 + 1 \times 1 = 26$$

$$AA = \begin{bmatrix} 34 & 20 \\ 20 & 26 \end{bmatrix}$$

3 Rewrite the full matrix multiplication, substituting the answer found in the previous part.

$$A^3 = AAA$$
$$= \begin{bmatrix} 34 & 20 \\ 20 & 26 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}$$

4 Calculate the second matrix multiplication (AAA).

$$AAA_{11} = 34 \times 3 + 20 \times 5 = 202$$
$$AAA_{21} = 34 \times 5 + 20 \times 1 = 190$$
$$AAA_{12} = 20 \times 3 + 26 \times 5 = 190$$
$$AAA_{22} = 20 \times 5 + 26 \times 1 = 126$$

$$AAA = \begin{bmatrix} 202 & 190 \\ 190 & 126 \end{bmatrix}$$

5 Write the answer.

$$A^3 = \begin{bmatrix} 202 & 190 \\ 190 & 126 \end{bmatrix}$$

EXERCISE 4.4 Matrix multiplication

PRACTISE

1 **WE7** Consider the matrix $C = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 4 & 6 \end{bmatrix}$. Evaluate the following.

a $4C$

b $\frac{1}{5}C$

c $0.3C$

2 Matrix D was multiplied by the scalar quantity x.

If $3D = \begin{bmatrix} 15 & 0 \\ 21 & 12 \\ 33 & 9 \end{bmatrix}$ and $xD = \begin{bmatrix} 12.5 & 0 \\ 17.5 & 10 \\ 27.5 & 7.5 \end{bmatrix}$, find the value of x.

3 **WE8** a If $X = [3 \quad 5]$ and $Y = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, show that the product matrix XY exists and state the order of XY.

b Determine which of the following matrices can be multiplied together and state the order of any product matrices that exist.

$$D = \begin{bmatrix} 7 & 4 \\ 3 & 5 \\ 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 5 & 7 \\ 8 & 9 \end{bmatrix} \text{ and } E = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 2 & 6 \end{bmatrix}$$

4 The product matrix ST has an order of 3×4. If matrix S has 2 columns, write down the order of matrices S and T.

5 WE9 a If $M = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $N = [7 \quad 12]$, determine the product matrix MN.

b Does the product matrix NM exist? Justify your answer by finding the product matrix NM and stating its order.

6 Matrix $S = [1 \quad 4 \quad 3]$, matrix $T = \begin{bmatrix} 2 \\ 3 \\ t \end{bmatrix}$ and the product matrix $ST = [5]$. Find the value of t.

7 WE10 For matrices $P = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$, determine the product matrix PQ.

8 For a concert, three different types of tickets can be purchased; adult, senior and child. The cost of each type of ticket is \$12.50, \$8.50 and \$6.00 respectively. The number of people attending the concert is shown in the following table.

Ticket type	Number of people
Adult	65
Senior	40
Child	85

a Construct a column matrix to represent the cost of the three different tickets in the order adult, senior and child.

If the number of people attending the concert is written as a row matrix, a matrix multiplication can be performed to determine the total amount in ticket sales for the concert.

b By finding the orders of each matrix and then the product matrix, explain why this is the case.

c By completing the matrix multiplication from part b, determine the total amount (in dollars) in ticket sales for the concert.

9 WE11 If $P = \begin{bmatrix} 8 & 2 \\ 4 & 7 \end{bmatrix}$, calculate the value of P^2.

10 If $T = \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$, calculate the value of T^3.

CONSOLIDATE

11 Consider the matrix $M = \begin{bmatrix} 12 & 9 & 15 \\ 36 & 6 & 21 \end{bmatrix}$. Which of the following is equal to the matrix M?

A $0.1\begin{bmatrix} 1.2 & 0.9 & 1.5 \\ 3.6 & 0.6 & 2.1 \end{bmatrix}$

B $3\begin{bmatrix} 3 & 3 & 5 \\ 9 & 2 & 7 \end{bmatrix}$

C $3\begin{bmatrix} 4 & 3 & 5 \\ 12 & 2 & 7 \end{bmatrix}$

D $3\begin{bmatrix} 36 & 27 & 45 \\ 108 & 18 & 63 \end{bmatrix}$

E $10\begin{bmatrix} 120 & 90 & 15 \\ 36 & 6 & 21 \end{bmatrix}$

12 Which of the following matrices can be multiplied together? Justify your answers by finding the order of the product matrices.

$$D = \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}, E = \begin{bmatrix} 5 & 8 \\ 7 & 1 \\ 9 & 3 \end{bmatrix}, F = \begin{bmatrix} 12 & 7 & 3 \\ 15 & 8 & 4 \end{bmatrix}, G = [13 \quad 15]$$

13 Find the product matrices when the following pairs of matrices are multiplied together.

a $[6 \quad 9]$ and $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ **b** $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and $[6 \quad 9]$ **c** $\begin{bmatrix} 7 \\ 2 \\ 9 \end{bmatrix}$ and $[10 \quad 15]$

d $\begin{bmatrix} 6 & 5 \\ 8 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 9 \end{bmatrix}$ **e** $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$

14 Evaluate the following matrix multiplications.

a $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

c Using your results from parts **a** and **b**, when will AB be equal to BA?

d If A and B are not of the same order, is it possible for AB to be equal to BA?

15 The 3×3 identity matrix, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

a Calculate the value of ${I_3}^2$. **b** Calculate the value of ${I_3}^3$.

c Calculate the value of ${I_3}^4$. **d** Comment on your answers to parts **a**–**c**.

16 The table below shows the percentage of students who are expected to be awarded grades A–E on their final examinations for Mathematics and Physics.

Grade	A	B	C	D	E
Percentage of students	5	18	45	25	7

The number of students studying Mathematics and Physics is 250 and 185 respectively.

a Construct a column matrix, S, to represent the number of students studying Mathematics and Physics.

b Construct a 1×5 matrix, A, to represent the percentage of students expected to receive each grade, expressing each element in decimal form.

c In the context of this problem, what does product matrix SA represent?

d Determine the product matrix SA. Write your answers correct to the nearest whole numbers.

e In the context of this problem, what does element SA_{12} represent?

17 A product matrix, $N = MPR$, has order 3×4. Matrix M has m rows and n columns, matrix P has order $1 \times q$, and matrix R has order $2 \times s$. Determine the values of m, n, s and q.

18 Dodgy Bros sell vans, utes and sedans. The average selling price for each type of vehicle is shown in the table below.

Type of vehicle	Monthly sales ($)
Vans	$4000
Utes	$12 500
Sedans	$8500

The table below shows the total number of vans, utes and sedans sold at Dodgy Bros in one month.

Type of vehicle	Number of sales
Vans	5
Utes	8
Sedans	4

Stan is the owner of Dodgy Bros and wants to determine the total amount of monthly sales.

a Explain how matrices could be used to help Stan determine the total amount, in dollars, of monthly sales.

b Perform a matrix multiplication that finds the total amount of monthly sales.

c Brian is Stan's brother and the accountant for Dodgy Bros. In finding the total amount of monthly sales, he performs the following matrix multiplication.

$$\begin{bmatrix} 5 \\ 8 \\ 4 \end{bmatrix} [4000 \quad 12\,500 \quad 8500]$$

Explain why this matrix multiplication is not valid for this problem.

19 Rhonda was asked to perform the following matrix multiplication to determine the product matrix GH.

$$GH = \begin{bmatrix} 6 & 5 \\ 3 & 8 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

Rhonda's answer was $\begin{bmatrix} 60 & 65 \\ 30 & 104 \\ 50 & 117 \end{bmatrix}$.

a By stating the order of product matrix GH, explain why Rhonda's answer is obviously incorrect.

b Determine the product matrix GH.

c Explain Rhonda's method of multiplying matrices and why this is the incorrect method.

d Provide simple steps to help Rhonda multiply matrices.

20 In an AFL game of football, 6 points are awarded for a goal and 1 point is awarded for a behind. St Kilda and Collingwood played in two grand finals in 2010, with the two results given by the following matrix multiplication.

$$\begin{bmatrix} 9 & 14 \\ 10 & 8 \\ 16 & 12 \\ 7 & 10 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 \\ S_1 \\ C_2 \\ S_2 \end{bmatrix}$$

Complete the matrix multiplication to determine the scores in the two grand finals.

MASTER

21 By using CAS or otherwise, calculate the following powers of square matrices.

a $\begin{bmatrix} 4 & 8 \\ 7 & 2 \end{bmatrix}^4$ b $\begin{bmatrix} \frac{2}{3} & \frac{5}{7} \\ \frac{1}{4} & 2 \end{bmatrix}^3$ c $\begin{bmatrix} 3 & 1 & 7 \\ 4 & 2 & 8 \\ 5 & 6 & 9 \end{bmatrix}^3$

22 The number of adults, children and seniors attending the zoo over Friday, Saturday and Sunday is shown in the table.

Day	Adults	Children	Seniors
Friday	125	245	89
Saturday	350	456	128
Sunday	421	523	102

Entry prices for adults, children and seniors are \$35, \$25, \$20 respectively.

a Using CAS or otherwise, perform a matrix multiplication that will find the entry fee collected for each of the three days.

b Write the calculation that finds the entry fee collected for Saturday.

c Is it possible to perform a matrix multiplication that would find the total for each type of entry fee (adults, children and seniors) over the three days? Explain your answer.

4.5 Inverse matrices and problem solving with matrices

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Inverse matrices

In the real number system, a number multiplied by its reciprocal results in 1. For example, $3 \times \frac{1}{3} = 1$. In this case $\frac{1}{3}$ is the reciprocal or multiplicative inverse of 3.

In matrices, if the product matrix is the identity matrix, then one of the matrices is the multiplicative inverse of the other.

For example,

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 3 + 5 \times -1 & 2 \times -5 + 5 \times 2 \\ 1 \times 3 + 3 \times -1 & 1 \times -5 + 3 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (the } 2 \times 2 \text{ identity matrix).}$$

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If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, then $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ is the multiplicative inverse of A, which is denoted as A^{-1}.

Similarly, if $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, then $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = B^{-1}$.

Hence $AA^{-1} = I = A^{-1}A$.

WORKED EXAMPLE 12

By finding the product matrix AB, determine whether the following matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

THINK	WRITE
1 Set up the product matrix.	$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$
2 Evaluate the product matrix.	$\begin{bmatrix} 2 \times 3 + 5 \times -1 & 2 \times -5 + 5 \times 2 \\ 1 \times 3 + 3 \times -1 & 1 \times -5 + 3 \times 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
3 Check if the product matrix is the identity matrix.	The product matrix AB is the identity matrix. Therefore, A and B are multiplicative inverses of each other.

Finding inverse matrices

Inverse matrices only exist for square matrices and can be easily found for matrices of order 2×2. Inverses can also be found for larger square matrices; however, the processes to find these are more complicated, so technology is often used to find larger inverses.

Remember that the product of a matrix and its inverse is the identity matrix, I.

For matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, let $A^{-1} = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$.

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so $$\begin{bmatrix} a \times e + c \times f & a \times g + c \times h \\ b \times e + d \times f & b \times g + d \times h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To find the values of e, f, g and h in terms of a, b, c and d would require four equations to be solved! However, there is a simpler method we can follow to find the inverse matrix A^{-1} in only three steps.

To determine the inverse for a matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$:

Step 1: **Swap the elements a_{11} and a_{22}: $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$**

Step 2: **Multiply elements a_{12} and a_{21} by -1: $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$**

Step 3: **Multiply by $\dfrac{1}{ad - bc}$:**

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

The expression $ad - bc$ is known as the **determinant** of matrix A. It is usually written as det A or $|A|$. If det $A = 0$, then the inverse matrix A^{-1} does not exist, because the value $\frac{1}{0}$ is undefined.

Note: In practice it is best to check that the determinant does not equal 0 before proceeding with the other steps.

WORKED EXAMPLE 13

If $A = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}$, determine A^{-1}.

THINK	WRITE
1 Swap elements a_{11} and a_{22}.	$\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$
2 Multiply elements a_{12} and a_{21} by -1.	$\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ -4 & 7 \end{bmatrix}$
3 Find the determinant (det $A = ad - bc$).	$a = 7, b = 2, c = 4, d = 1$ $ad - bc = 7 \times 1 - 2 \times 4$ $= -1$
4 Multiply the matrix from step 2 by $\frac{1}{\det A}$.	$\frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -4 & 7 \end{bmatrix} = -1\begin{bmatrix} 1 & -2 \\ -4 & 7 \end{bmatrix}$ $= \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$

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Using inverse matrices to solve problems

Unlike in the real number system, we can't divide one matrix by another matrix. However, we can use inverse matrices to help us solve matrix equations in the same way that division is used to help solve many linear equations.

Given the matrix equation $AX = B$, the inverse matrix can be used to find the matrix X as follows.

Step 1: (Multiply both sides of the equation by A^{-1}): $A^{-1}AX = A^{-1}B$

Step 2: ($A^{-1}A = I$, the identity matrix): $IX = A^{-1}B$

Step 3: ($IX = X$, as found in the previous section): $X = A^{-1}B$

Note: If we multiply the left-hand side of our equation by A^{-1} on the left, then we must also multiply the right-hand side of our equation by A^{-1} on the left.

Remember that when multiplying with matrices the order of the multiplication is important.

If the equation was $XA = B$ and matrix X needed to be found, then the inverse matrix multiplication would be:

Step 1: $XAA^{-1} = BA^{-1}$

Step 2: $XI = BA^{-1}$

Step 3: $X = BA^{-1}$

WORKED EXAMPLE 14

If $\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$, find the values of x and y.

THINK	WRITE
1 The matrix equation is in the form $AX = B$. Identify matrices A, B and X.	$A = \begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$
2 To determine X we need to multiply both sides of the equation by the inverse A^{-1}.	$A^{-1}AX = A^{-1}B$
3 Find the inverse A^{-1}.	$A^{-1} = \dfrac{1}{12 - 10}\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ $= \dfrac{1}{2}\begin{bmatrix} 3 & -1 \\ -5 & 4 \end{bmatrix}$ $= \begin{bmatrix} 1.5 & 2 \\ -2.5 & 2 \end{bmatrix}$
4 Calculate $A^{-1}B$.	$A^{-1} = \begin{bmatrix} 1.5 & 1 \\ -2.5 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix}$ $= \begin{bmatrix} 1.5 \times 8 + -1 \times 11 \\ -2.5 \times 8 + 2 \times 11 \end{bmatrix}$ $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
5 Solve for x and y.	$X = A^{-1}B$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $x = 1$ and $y = 2$

Using inverse matrices to solve a system of simultaneous equations

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Interactivity
Using matrices to solve simultaneous equations
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If you have a pair of simultaneous equations, they can be set up as a matrix equation and solved using inverse matrices.

Take the pair of simultaneous equations $ax + by = c$ and $dx + ey = f$.

These can be set up as the matrix equation

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}.$$

If we let $A = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c \\ f \end{bmatrix}$, this equation is of the form $AX = B$, which can be solved as $X = A^{-1}B$ (as determined previously).

WORKED EXAMPLE 15

Solve the following pair of simultaneous equations by using inverse matrices.

$$2x + 3y = 6$$
$$4x - 6y = -4$$

THINK

1 Set up the simultaneous equations as a matrix equation.

WRITE

$$\begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

2 Find the inverse of the matrix A, A^{-1}.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-12-12}\begin{bmatrix} -6 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \frac{1}{-24}\begin{bmatrix} -6 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & -\frac{1}{12} \end{bmatrix}$$

3 Calculate $A^{-1}B$.

$$A^{-1}B = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & -\frac{1}{12} \end{bmatrix}\begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} \times 6 + \frac{1}{8} \times -4 \\ \frac{1}{6} \times 6 + -\frac{1}{12} \times -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$$

4 State the answer.

$$x = 1,\ y = \frac{4}{3}$$

Using matrix equations to solve worded problems

To use matrices to solve worded problems, you must set up a matrix equation from the information provided. The matrix equation can then be solved using the skills you have previously learned.

WORKED EXAMPLE 16

On an excursion, a group of students and teachers travelled to the city by train and returned by bus. On the train. the cost of a student ticket was \$3 and the cost of a teacher ticket was \$4.50, with the total cost for the train tickets being \$148.50. On the bus, the cost of a student ticket was \$2.75 and the cost of a teacher ticket was \$3.95, with the total cost for the bus tickets being \$135.60.

By solving a matrix equation, determine how many students and teachers attended the excursion.

THINK	WRITE
1 Identify the two unknowns in the problem. Assign a pronumeral to represent each unknown.	Numbers of students $= s$, number of teachers $= t$
2 Construct a matrix to represent the unknowns.	$\begin{bmatrix} s \\ t \end{bmatrix}$
3 Highlight the key information, that is, how much the two different types of tickets were for students and teachers.	Student train ticket = \$3 Teacher train ticket = \$4.50 Student bus ticket = \$2.75 Teacher bus ticket = \$3.95
4 Construct a matrix to represent the information. Note that each row represents the two different types of travel.	$\begin{bmatrix} 3 & 4.50 \\ 2.75 & 3.95 \end{bmatrix}\begin{matrix}\text{Train} \\ \text{Bus}\end{matrix}$
5 Construct a matrix to represent the total cost in the same row order as in step 4.	$\begin{bmatrix} 148.50 \\ 135.60 \end{bmatrix}$
6 Set up a matrix equation in the form $AX = B$, remembering that X will represent the 'unknowns', that is, the values that need to be found.	$\begin{bmatrix} 3 & 4.50 \\ 2.75 & 3.95 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 148.50 \\ 135.60 \end{bmatrix}$
7 Solve the matrix equation by finding A^{-1} and multiplying it by B.	$A^{-1} = \dfrac{1}{3 \times 3.95 - 2.75 \times 4.50}\begin{bmatrix} 3.95 & -4.50 \\ -2.75 & 3 \end{bmatrix}$ $= \dfrac{1}{-0.525}\begin{bmatrix} 3.95 & -4.50 \\ -2.75 & 3 \end{bmatrix}$ $\begin{bmatrix} s \\ t \end{bmatrix} = A^{-1}B$ $= \dfrac{1}{-0.525}\begin{bmatrix} 3.95 & -4.50 \\ -2.75 & 3 \end{bmatrix}\begin{bmatrix} 148.50 \\ 135.60 \end{bmatrix}$ $= \dfrac{1}{-0.525}\begin{bmatrix} -23.625 \\ -1.575 \end{bmatrix}$ $= \begin{bmatrix} 45 \\ 3 \end{bmatrix}$
8 Answer the question.	There were 45 students and 3 teachers on the excursion.

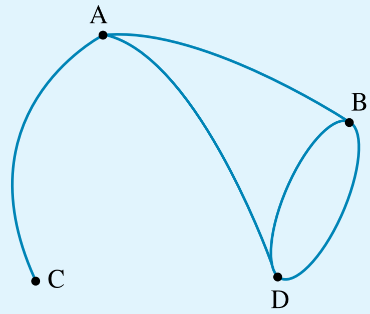
Adjacency matrices

Matrices can be used to determine the number of different connections between objects, such as towns or people. They can also be used to represent tournament outcomes and determine overall winners. To determine the number of connections between objects, a matrix known as an **adjacency matrix** is set up to represent these connections.

WORKED EXAMPLE 17

The diagram at right shows the number of roads connecting between four towns, A, B, C and D.

Construct an adjacency matrix to represent this information.



THINK

1 Since there are four connecting towns, a 4×4 adjacency matrix needs to be constructed. Label the row and columns with the relevant towns A, B, C and D.

WRITE

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \end{array}$$

2 There is one road connecting town A to town B, so enter 1 in the cell from A to B.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} - & - & - & - \\ \textcircled{1} & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \end{array}$$

3 There is also only one road between town A and towns C and D; therefore, enter 1 in the appropriate matrix positions. There are no loops at town A (i.e. a road connecting A to A); therefore, enter 0 in this position.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & - & - & - \\ 1 & - & - & - \\ 1 & - & - & - \\ 1 & - & - & - \end{bmatrix} \end{array}$$

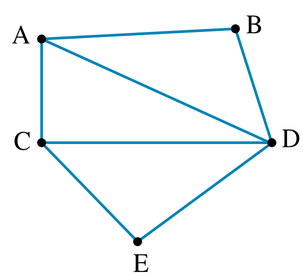
4 Repeat this process for towns B, C and D. Note that there are two roads connecting towns B and D, and that town C only connects to town A.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

Determining the number of connections between objects

An adjacency matrix allows us to determine the number of connections either directly between or via objects. If a direct connection between two objects is denoted as one 'step', 'two steps' means a connection between two objects via a third object, for example the number of ways a person can travel between towns A and D via another town.

You can determine the number of connections of differing 'steps' by raising the adjacency matrix to the power that reflects the number of steps in the connection.



For example, the following diagram shows the number of roads connecting five towns, A,B, C, D and E. There are a number of ways to travel between towns A and D. There is one direct path between the towns; this is a one-step path. However, you can also travel between towns A and D via town C or B. These are considered two-step paths as there are two links (or roads) in these paths. The power on the adjacency matrix would therefore be 2 in this case.

WORKED EXAMPLE 18

The following adjacency matrix shows the number of pathways between four attractions at the zoo: lions (L), seals (S), monkeys (M) and elephants (E).

$$\begin{array}{c} \\ L \\ S \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cccc} L & S & M & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Using CAS or otherwise, determine how many ways a family can travel from the lions to the monkeys via one of the other two attractions.

THINK	WRITE
1 Determine the link length.	The required path is between two attractions via a third attraction, so the link length is 2.
2 Using CAS or otherwise, evaluate the matrix	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 \\ 3 & 3 & 6 & 2 \\ 3 & 3 & 2 & 6 \end{bmatrix}$
3 Interpret the information in the matrix and answer the question by locating the required value.	$\begin{bmatrix} 3 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 \\ 3 & 3 & 6 & \textcircled{2} \\ 3 & 3 & 2 & 6 \end{bmatrix}$ There are 3 ways in which a family can travel from the lions to the monkeys via one of the other two attractions.

EXERCISE 4.5 Inverse matrices and problem solving with matrices

PRACTISE

1 **WE12** By finding the product matrix AB, determine whether the following matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1.5 & -2.5 \\ -1 & 2 \end{bmatrix}$$

2 Matrices A and B are multiplicative inverses of each other.

$$\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ a & 1 \end{bmatrix}$$

By finding the product matrix AB, show that the value of $a = -2$.

3 **WE13** Find the inverses of the following matrices:

a $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$

4 Consider the matrix $B = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$. By finding the value of the determinant, explain why B^{-1} does not exist.

5 **WE14** If $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$, find the values of x and y.

6 A matrix equation is represented by $XA = B$, where $B = [1 \quad 2]$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1.5 & 2 \end{bmatrix}$.

a State the order of matrix X and hence find matrix X.

b Find matrix A.

7 **WE15** Solve the following pair of simultaneous equations by using inverse matrices.

$$x + 2y = 4$$
$$3x - 5y = 1$$

8 Show that there is no solution to the following pair of simultaneous equations by attempting to solve them using inverse matrices.

$$3x + 5y = 4$$
$$4.5x + 7.5y = 5$$

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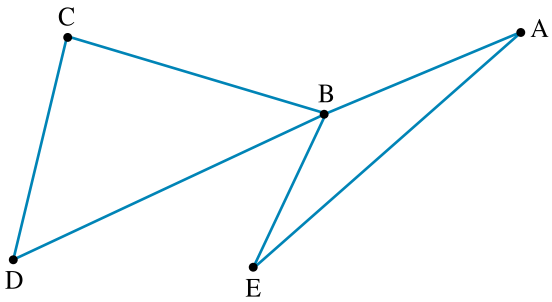
9 **WE16** For his 8th birthday party, Ben and his friends went ice skating and ten-pin bowling. The price for ice skating was \$4.50 per child and \$6.50 per adult, with the total cost for the ice skating being \$51. For the ten-pin bowling, the children were charged \$3.25 each and the adults were charged \$4.95 each, with the total cost for the bowling being \$37.60. By solving a matrix equation, determine how many children (including Ben) attended the party.

10 At the cinema, Justine and her friends bought 5 drinks and 4 bags of popcorn, spending \$14. Sarah and her friends bought 4 drinks and 3 bags of popcorn, spending \$10.80. By solving a matrix equation, determine the price of 2 drinks and 2 bags of popcorn.

11 **WE17** The diagram below shows the network cable between five main computers (A, B, C D and E) in an office building.



Construct an adjacency matrix to represent this information.

12 There are five friends on a social media site: Peta, Seth, Tran, Ned and Wen. The number of communications made between these friends in the last 24 hours is shown in the adjacency matrix below.

$$\begin{array}{c} \\ P \\ S \\ T \\ N \\ W \end{array} \begin{array}{c} \begin{array}{ccccc} P & S & T & N & W \end{array} \\ \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 & 4 \\ 3 & 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

a How many times did Peta and Tran communicate over the last 24 hours?
b Did Seth communicate with Ned at any time during the last 24 hours?
c In the context of this problem, explain the existence of the zeros along the diagonal.
d Using the adjacency matrix, construct a diagram that shows the number of communications between the five friends.

13 WE18 The adjacency matrix below shows the number of roads between three country towns, Glenorchy (G), St Arnaud (S) and Campbells Bridge (C).

$$\begin{array}{c} \\ G \\ S \\ C \end{array} \begin{array}{c} \begin{array}{ccc} G & S & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \end{array}$$

Using CAS or otherwise, determine the number of ways a person can travel from Glenorchy to St Arnaud via Campbell's Bridge.

14 The direct Cape Air flights between five cities, Boston (B), Hyannis (H), Martha's Vineyard (M), Nantucket (N) and Providence (P), are shown in the adjacency matrix.

$$\begin{array}{c} \\ B \\ H \\ M \\ N \\ P \end{array} \begin{array}{c} \begin{array}{ccccc} B & H & M & N & P \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a Construct a diagram to represent the direct flights between the five cities.
b Construct a matrix that determines the number of ways a person can fly between two cities via another city.
c Explain how you would determine the number of ways a person can fly between two cities via two other cities.
d Is it possible to fly from Boston and stop at every other city? Explain how you would answer this question.

CONSOLIDATE

15 Consider the matrix equation $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$.

a Explain how this matrix equation can be solved using the inverse matrix.
b State the inverse matrix used to solve this matrix equation.
c Calculate the values of x and y, clearly showing your working.

16 Veronica bought two donuts and three cupcakes for $14. The next week she bought three donuts and two cupcakes for $12.25. This information is shown in the matrices below, where d and c represent the cost of a donut and cupcake respectively.

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} d \\ c \end{bmatrix} = \begin{bmatrix} 14.00 \\ 12.25 \end{bmatrix}$$

By solving the matrix equation, determine how much Veronica would pay for four donuts and three cupcakes.

17 Jeremy has an interest in making jewellery, and he makes bracelets and necklaces which he sells to his friends. He charges the same amount for each bracelet and necklace, regardless of the quantity sold.

Johanna buys 3 bracelets and 2 necklaces from Jeremy for $31.80.

Mystique buys 5 bracelets and 3 necklaces from Jeremy for $49.80.

a Construct a pair of simultaneous equations and use an inverse matrix to help determine the prices that Jeremy charges for each bracelet and each necklace.
b How much would Jeremy charge for 7 bracelets and 4 necklaces?

18 a Find the determinants of the following matrices to determine which of them have inverses.

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}, C = \begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix}, D = \begin{bmatrix} 0 & 1 \\ 3 & -5 \end{bmatrix}$$

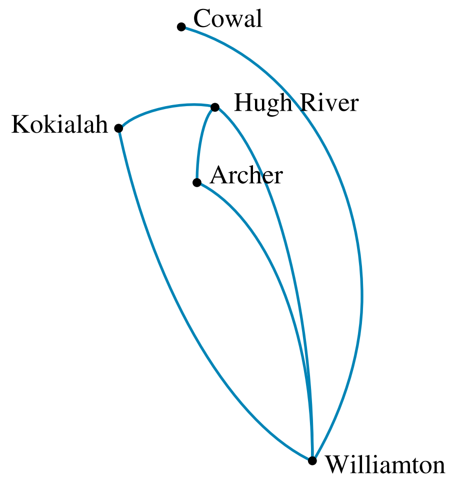
b For those matrices that have inverses, find the inverse matrices.

19 The adjacency matrix below shows the number of text messages sent between three friends, Stacey (S), Ruth (R) and Toiya (T), immediately after school one day

$$\begin{array}{c} \\ S \\ R \\ T \end{array}\begin{array}{c} \begin{array}{ccc} S & R & T \end{array} \\ \begin{bmatrix} 0 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

a State the number of text messages sent between Stacey and Ruth.
b Determine the total number of text messages sent between all three friends.

20 Airlink flies charter flights in the Cape Lancaster region. The direct flights between Williamton, Cowal, Hugh River, Kokialah and Archer are shown in the diagram.



a Using the diagram, construct an adjacency matrix that shows the number of direct flights between the five towns

b How many ways can a person travel between Williamton and Kokialah via another town?

c Is it possible to fly between Cowal and Archer and stop over at two other towns? Justify your answer.

21 Stefan was asked to solve the following matrix equation.

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 9 \end{bmatrix}$$

His first step was to evaluate $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}^{-1}$. Stefan wrote $6\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, which was incorrect.

a Explain one of the errors Stefan made in finding the inverse matrix.

b Hence, find the correct inverse matrix $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}^{-1}$.

Stefan's next step was to perform the following matrix multiplication.

$$6\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 9 \end{bmatrix}$$

c By finding the order of both matrices, explain why this multiplication is not possible.

d Write the steps Stefan should have used to calculate this matrix multiplication.

e Using your steps from part **d**, determine the values of x and y.

22 WholeFoods distribute two different types of apples, Sundowners and Pink Ladies, to two supermarkets, Foodsale and Betafoods. Foodsale orders 5 boxes of Sundowners and 7 boxes of Pink Ladies, with their order totalling \$156.80. Betafoods pays \$155.40 for 6 boxes each of Sundowners and Pink Ladies.

The matrix below represents part of this information, where s and p represent the price for a box of Sundowners and Pink Ladies respectively.

$$\begin{bmatrix} 5 & \\ 6 & 6 \end{bmatrix}\begin{bmatrix} s \\ p \end{bmatrix} = \begin{bmatrix} 156.80 \\ \end{bmatrix}$$

a Complete the matrix equation.

b By solving the matrix equation using CAS or otherwise, determine the cost of a box of Sundowner apples.

There are 5 kg of apples in each box. Betafoods sells Sundowner apples for \$3.49 per kilogram and Pink Ladies for \$4.50 per kilogram.

c Construct a row matrix, K, to represent the number of kilograms of Sundowners and Pink Ladies in Betafood's order.

A matrix representing the selling price, S, of each type of apple is constructed. A matrix multiplication is performed that determines the total selling price in dollars, for both types of apples.

d Write the order of matrix S.

e By performing the matrix multiplication, determine the total amount (in dollars) in revenue if all the apples are sold at the price stated.

f Determine the profit, in dollars, made by Betafoods if all apples are sold at the stated selling prices.

23 Four hundred tickets were sold for the opening of the movie *The Robbit* at the Dendy Cinema. Two types of tickets were sold: adult and concession. Adult tickets were \$15.00 and concession tickets were \$9.50. The total revenue from the ticket sales was \$5422.50.

a Identify the two unknowns and construct a pair of simultaneous equations to represent this information.

b Set up a matrix equation representing this information.

c Using CAS or otherwise, determine the number of adult tickets sold.

24 The senior school manager developed a matrix formula to determine the number of school jackets to order for Years 11 and 12 students. The column matrix, J_0, shows the number of jackets ordered last year.

$$J_0 = \begin{bmatrix} 250 \\ 295 \end{bmatrix}$$

J_1 is the column matrix that lists the number of Year 11 and 12 jackets to be ordered this year. J_1 is given by the matrix formula

$$J_1 = AJ_0 + B, \text{ where } A = \begin{bmatrix} 0.65 & 0 \\ 0 & 0.82 \end{bmatrix} \text{ and } B = \begin{bmatrix} 13 \\ 19 \end{bmatrix}.$$

a Using CAS or otherwise, determine J_1.

b Using your value from part **a** and the same matrix formula, determine the jacket order for the next year. Write your answer to the nearest whole number.

MASTER

25 Using CAS, find the inverse of the following matrices.

a $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 1 & -1 & -2 \\ -1 & 2 & 0 & 2 \\ 0 & 3 & 5 & 3 \\ 1 & 1 & 4 & 1 \end{bmatrix}$

26 Using CAS, solve the following matrix equation to find the values of a, b, c and d.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & -1 & 3 \\ 3 & 1 & 1 & -2 \\ -1 & 2 & -4 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 12 \\ -14 \end{bmatrix}$$

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4.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

To access eBookPLUS activities, log on to

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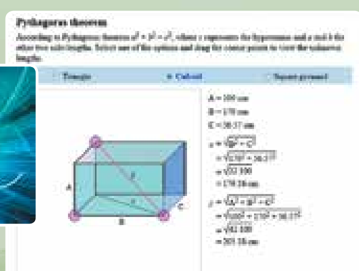
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.



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Units 1 & 2 | Matrices

Sit topic test

4 Answers

EXERCISE 4.2

1 $\begin{bmatrix} 18 & 12 & 8 \\ 13 & 10 & 11 \end{bmatrix}$

2 $\begin{bmatrix} 6 & 4 & 7 & 3 & 6 \\ 2 & 6 & 6 & 6 & 4 \end{bmatrix}$

3 a 545 km

b Coober Pedy and Alice Springs

c 2458 km

4 a $\begin{array}{c} \\ M \\ C \\ S \end{array}\begin{array}{c} \begin{array}{ccc} M & C & S \end{array} \\ \begin{bmatrix} 0 & 91.13 & 110.72 \\ 91.13 & 0 & 48.02 \\ 110.72 & 48.02 & 0 \end{bmatrix} \end{array}$

b \$139.15

5 $\begin{bmatrix} 45 \\ 30 \end{bmatrix}$, order 2×1

6 a 56 **b** 213 **c** $\begin{bmatrix} 12 & 17 & 18 \end{bmatrix}$

7 a 5 **b** 6 **c** 7

8 a There is no 4th column.

b e_{23}

c Nadia thought that e_{12} was read as 1st column, 2nd row. The correct value is 0.

9 A: 1×1, B 3×1, C 1×2

10 a **a** and **d** are matrices with orders of 2×1 and 2×4 respectively. The matrix shown in **b** is incomplete, and the matrix shown in **c** has a different number of rows in each column.

11 a 3 **b** −1 **c** 1 **d** 0.5 **e** 0.9

12 a $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

13 a 3×2 **b** $\begin{bmatrix} 4 & 3 \\ -1 & 7 \\ -4 & 6 \end{bmatrix}$

14 a $\begin{bmatrix} 0.3 \\ 4.2 \\ 6.8 \\ 0.2 \\ 1.6 \\ 2.1 \\ 0.5 \\ 5.2 \end{bmatrix}$

b $\begin{bmatrix} 2358 & 68\,330 & 227\,600 & 801\,428 & 984\,000 & 1\,346\,200 & 1\,727\,200 & 2\,529\,875 \end{bmatrix}$

c i 3×2 **ii** $\begin{bmatrix} 801\,428 & 6.8 \\ 227\,600 & 5.2 \\ 1\,727\,200 & 4.2 \end{bmatrix}$

15 a $\begin{bmatrix} 148\,178 & 2.2 \\ 30\,839 & 0.6 \\ 146\,429 & 3.6 \\ 26\,044 & 1.7 \\ 77\,928 & 3.8 \\ 16\,900 & 3.4 \\ 66\,582 & 31.6 \\ 4\,043 & 1.2 \end{bmatrix}$

b i 66 582 **ii** 16 900 **iii** 325 446

c 516 943

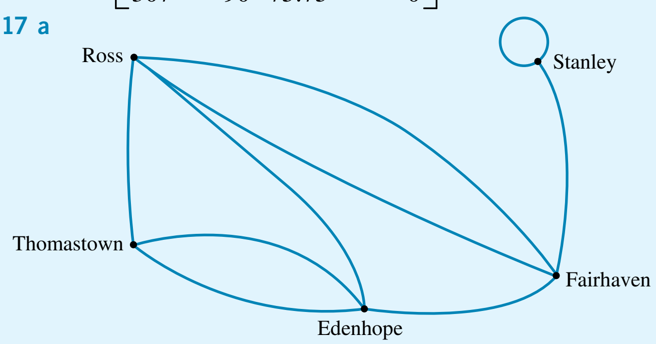
16 a The zeros mean they don't fly from one place back to the same place.

b \$175

c Mount Isa

d $\begin{array}{c} \\ O \\ B \\ D \\ M \end{array}\begin{array}{c} \begin{array}{cccc} O & B & D & M \end{array} \\ \begin{bmatrix} 0 & 59.50 & 150 & 190 \\ 89 & 0 & 85 & 75 \\ 175 & 205 & 0 & 213.75 \\ 307 & 90 & 75.75 & 0 \end{bmatrix} \end{array}$

17 a



b i True **ii** False

iii True **iv** False

c N_{31} and N_{13}

18 a D

b

Question	1	2	3	4	5	6	7	8	9	10
Response	A	D	B	E	D	B	E	D	B	E

c There are no 1s in row A, just 0s.

d

$$\text{Next question}\ \begin{matrix} \\ A \\ B \\ C \\ D \\ E \end{matrix}\ \overset{\text{This question}}{\begin{matrix} A & B & C & D & E \end{matrix}}\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

19 Check with your teacher.

20 a Check with your teacher. Possible answers:

i C1 **ii** B2 **iii** B3 **iv** A2

b i The column position corresponds to the alphabetical order, e.g. column 2 would be B, the second letter in the alphabet. The row position corresponds to the cell row, e.g. row 4 would be cell row 4.

ii The cell number would be the mth letter in the alphabet and the nth row.

EXERCISE 4.3

1 a $\begin{bmatrix} 1 & 6 \\ -1 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -0.1 \\ 1.5 \\ 4.2 \end{bmatrix}$

2 $a = 4, b = -6$

3 $\begin{bmatrix} -1 & -2 \\ -1 & -5 \end{bmatrix}$

4 a Both matrices must be of the same order for it to be possible to add and subtract them.

b $B = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$

5 a $\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$ **b** $\begin{bmatrix} -5 \\ 3 \\ 6 \end{bmatrix}$ **c** $\begin{bmatrix} 6 \\ 4 \\ -6 \end{bmatrix}$ **d** $\begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix}$

6 a $[-0.25\ \ -0.95\ \ 0.3]$ **b** $\begin{bmatrix} 3 & -1 \\ 9 & 1 \end{bmatrix}$

c $\begin{bmatrix} -1 & 5 & 1 \\ 4 & -2 & 3 \\ 3 & -3 & -2 \end{bmatrix}$ **d** $\begin{bmatrix} 15 & 15 & 7 \\ 9 & 10 & 8 \end{bmatrix}$

7 $a = -5, b = 2, c = 5$

8 $a = 6, b = -1, c = 5$

9 A and E have the same order, 1×2.

B and C have the same order, 2×1.

10 a $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 3 \end{bmatrix}$

b i $\begin{bmatrix} 3 & 4 & 8 \\ 6 & 8 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 3 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \end{bmatrix}$

Eggs	Small	Medium	Large
Free range	1	1	3
Barn laid	2	2	2

11 a Both matrices are of the order 2×3; therefore, the answer matrix must also be of the order 2×3. Marco's answer matrix is of the order 2×1, which is incorrect.

b Check with your teacher. A possible response is:

Step 1: Check that all matrices are the same order.

Step 2: Add or subtract the corresponding elements.

12 a $\begin{bmatrix} 150 \\ 165 \\ 155 \\ 80 \end{bmatrix}$

b i $\begin{bmatrix} 150 \\ 165 \\ 155 \\ 80 \end{bmatrix} + \begin{bmatrix} 145 \\ 152 \\ 135 \\ 95 \end{bmatrix} + \begin{bmatrix} 166 \\ 155 \\ 156 \\ 110 \end{bmatrix}$

ii $\begin{bmatrix} 461 \\ 472 \\ 446 \\ 285 \end{bmatrix}$

c $\begin{bmatrix} 35 \\ 41 \\ 38 \\ 25 \end{bmatrix} + \begin{bmatrix} 32 \\ 36 \\ 35 \\ 30 \end{bmatrix} + \begin{bmatrix} 38 \\ 35 \\ 35 \\ 32 \end{bmatrix} = \begin{bmatrix} 105 \\ 112 \\ 108 \\ 87 \end{bmatrix}$

13 a $\begin{bmatrix} 12 \\ 9 \\ 8 \end{bmatrix} + \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$ or $\begin{bmatrix} 12 \\ 9 \\ 8 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

b $\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ 10 \end{bmatrix}$ or $\begin{bmatrix} 12 \\ 4 \\ 10 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

14 a To add or subtract matrices, all matrices must be of the same order. Since the resultant matrix D is of the order 3×2, all other matrices must also be of the order 3×2.

b $\begin{bmatrix} 2x - 12 & y - 2 \\ 2x + 10 & 5 - y \\ 3x + 10 & -2 - 2x \end{bmatrix}$

c Check with your teacher.

15 $\begin{bmatrix} \frac{5}{8} & \frac{5}{8} & 0 \\ \frac{17}{30} & \frac{5}{42} & \frac{-5}{9} \\ \frac{11}{18} & \frac{1}{8} & \frac{-2}{9} \end{bmatrix}$

16 a Check with your teacher. Possible answer:
= sum (A1 + D1) = sum (B1 + E1) = sum (C1 + F1)
= sum (A2 + D2) = sum (B2 + E2) = sum (C2 + F2)

b $\begin{bmatrix} 11 & 29 & 20 \\ 54 & 5 & 27 \end{bmatrix}$

EXERCISE 4.4

1 a $\begin{bmatrix} 8 & 12 & 28 \\ 4 & 16 & 24 \end{bmatrix}$

b $\begin{bmatrix} \frac{2}{5} & \frac{3}{5} & \frac{7}{5} \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} \end{bmatrix}$

c $\begin{bmatrix} 0.6 & 0.9 & 2.1 \\ 0.3 & 1.2 & 1.8 \end{bmatrix}$

2 $x = 2.5$

3 a $1 \times 2\ 2 \times 1$

Number of columns = number of rows, therefore XY exists and is of order 1×1.

b DE: 3×3, DC: 3×2, ED: 2×2, CE: 2×3

4 S: 3×2, T: 2×4

5 a $MN = \begin{bmatrix} 28 & 48 \\ 21 & 36 \end{bmatrix}$

b Yes, [64] is of the order 1×1.

6 $t = -3$

7 $PQ = \begin{bmatrix} 41 & 45 \\ 36 & 32 \end{bmatrix}$

8 a $\begin{bmatrix} 12.50 \\ 8.50 \\ 6.00 \end{bmatrix}$

b Total tickets requires an order of 1×1, and the order of the ticket price is 3×1. The number of people must be of order 1×3 to result in a product matrix of order 1×1. Therefore, the answer must be a row matrix.

c $1662.50

9 $\begin{bmatrix} 72 & 30 \\ 60 & 57 \end{bmatrix}$

10 $\begin{bmatrix} 27 & 315 \\ 0 & 216 \end{bmatrix}$

11 C

12 DG: 3×2, FD: 2×1, FE: 2×2, EF: 3×3, GF: 1×3

13 a [66]

b $\begin{bmatrix} 30 & 45 \\ 24 & 36 \end{bmatrix}$

c $\begin{bmatrix} 70 & 105 \\ 20 & 30 \\ 90 & 135 \end{bmatrix}$

d $\begin{bmatrix} 57 \\ 43 \end{bmatrix}$

e $\begin{bmatrix} 38 & 19 \\ 14 & 7 \end{bmatrix}$

14 a $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

b $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

c When either A or B is the identity matrix

d No. Consider matrix A with order of $m \times n$ and matrix B with order of $p \times q$, where $m \neq p$ and $n \neq q$.
If AB exists, then it has order $m \times q$ and $n = p$.
If BA exists, then it has order $p \times n$ and $q = m$.
Therefore $AB \neq BA$, unless $m = p$ and $n = q$, which is not possible since they are of different orders.

15 a $I_3{}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b $I_3{}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c $I_3{}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d Whatever power you raise I to, the matrix stays the same.

16 a $\begin{bmatrix} 250 \\ 185 \end{bmatrix}$

b [0.05 0.18 0.45 0.25 0.07]

c The number of expected grades (A–E) for students studying Mathematics and Physics.

d $\begin{bmatrix} 13 & 45 & 113 & 63 & 18 \\ 9 & 33 & 83 & 46 & 13 \end{bmatrix}$

e 45 students studying maths are expected to be awarded a B grade.

17 $n = 1$, $m = 3$, $s = 4$, $q = 2$

18 a Check with your teacher. Possible answer:

Represent the number of vehicles in a row matrix and the cost for each vehicle in a column matrix, then multiply the two matrices together. The product matrix will have an order of 1×1.

b [154 000] or $154 000

c Check with your teacher. Possible answer:

In this multiplication each vehicle is multiplied by price of each type of vehicle, which is incorrect. For example, the ute is valued at $12 500, but in this multiplication the eight utes sold are multiplied by $4000, $12 500 and $8500 respectively.

19 a Matrix G is of order 3×2 and matrix H is of order 2×1; therefore, GH is of order 3×1. Rhonda's matrix has an order of 3×2.

b $\begin{bmatrix} 125 \\ 134 \\ 167 \end{bmatrix}$

c Check with your teacher. Possible answer:

Rhonda multiplied the first column with the first row, and then the second column with the second row.

d Check with your teacher. Possible answer:

Step 1: Find the order of the product matrix.

Step 2: Multiply the elements in the first row by the elements in the first column.

20 $C_1 = 68$, $S_1 = 68$, $C_2 = 108$, $S_2 = 52$. The two results were 68–68 and 108–52.

21 a $\begin{bmatrix} 4 & 8 \\ 7 & 2 \end{bmatrix}^4 = \begin{bmatrix} 7200 & 6336 \\ 5544 & 5616 \end{bmatrix}$

b $\begin{bmatrix} \frac{2}{3} & \frac{5}{7} \\ \frac{1}{4} & 2 \end{bmatrix}^3 = \begin{bmatrix} \frac{337}{378} & \frac{7505}{1764} \\ \frac{1501}{1008} & \frac{53}{6} \end{bmatrix}$

c $\begin{bmatrix} 3 & 1 & 7 \\ 4 & 2 & 8 \\ 5 & 6 & 9 \end{bmatrix}^3 = \begin{bmatrix} 792 & 694 & 1540 \\ 984 & 868 & 1912 \\ 1356 & 1210 & 2632 \end{bmatrix}$

22 a $\begin{bmatrix} 12\,280 \\ 26\,210 \\ 29\,850 \end{bmatrix}$; Friday \$12 280, Saturday \$26 210, Sunday \$29 850

b $350 \times 35 + 456 \times 25 + 128 \times 20$

c No, because you cannot multiply the entry price (3×1) by the number of people (3×3).

EXERCISE 4.5

1 Yes, $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2 $AB = \begin{bmatrix} 3 + a & 0 \\ 6 + 3a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3 a $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ -1.5 & 3.5 \end{bmatrix}$

c $\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$

4 det $B = 0$. We cannot divide by zero; therefore, B^{-1} does not exist.

5 $x = 2, y = -3$

6 a Matrix X has an order of 1×2. $X = [-2 \;\; 3]$

b $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$

7 $x = 2, y = 1$

8 The determinant = 0, so no inverse exists. This means that there is no solution to the simultaneous equations.

9 7 children and 3 adults

10 \$6.40

11

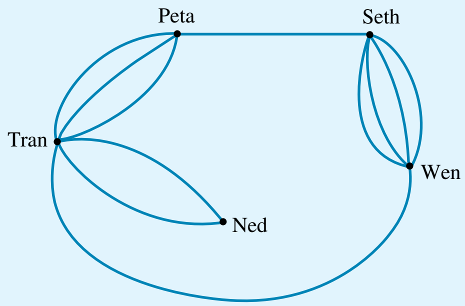
	A	B	C	D	E
A	0	1	0	0	1
B	1	0	1	1	1
C	0	1	0	1	0
D	0	1	1	0	0
E	1	1	0	0	0

12 a 3

b No

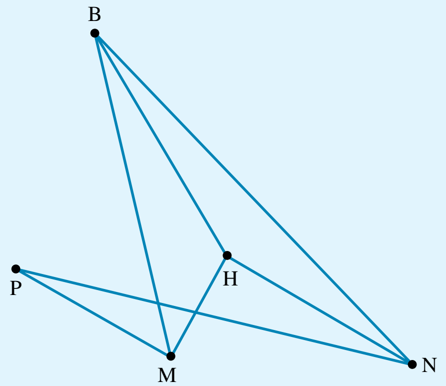
c They did not communicate with themselves.

d



13 2

14 a



b

	B	H	M	N	P
B	3	2	2	2	2
H	2	3	2	2	2
M	2	2	4	3	1
N	2	2	3	4	1
P	2	2	1	1	2

c Raise the matrix to a power of 2.

d Yes. Raise the matrix to a power of 4, as there are five cities in total.

15 a Find the inverse of $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ and multiply by $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$, using $AX = B$ and $X = A^{-1}B$.

b $\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$

c $x = 2, y = 1$

16 \$17.50

17 a Each bracelet costs \$4.20 and each necklace costs \$9.60.

b \$67.80

18 a det $A = 1$, det $B = 0$, det $C = 0$, det $D = -3$. Therefore, matrices A and D have inverses.

b $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$, $D^{-1} = \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ 1 & 0 \end{bmatrix}$

19 a 3

b 6

20 a

$$\begin{array}{c} \\ W \\ C \\ H \\ K \\ A \end{array}\begin{array}{c} \begin{array}{ccccc} W & C & H & K & A \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b 1

c Yes. The matrix raised to the power of 3 will provide the number of ways possible.

21 a Any one of: did not swap the elements on the diagonal; did not multiply the other elements by −1; or did not multiply the matrix by $\frac{1}{\det}$.

b $\frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$

c The respective order of matrices is 2×2 and 1×2. The number of columns in the first matrix does not equal the number of rows in the second matrix.

d Check with your teacher. Possible answers:

Step 1: Find the correct inverse.

Step 2: Multiply $[5 \;\; 9]\frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$.

e $x = 1, y = 3$

22 a $\begin{bmatrix} 5 & 7 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} s \\ p \end{bmatrix} = \begin{bmatrix} 156.80 \\ 155.40 \end{bmatrix}$

b \$12.25

c $K = [30 \;\; 30]$

d 2×1

e \$239.70

f \$84.30

23 a a = number of adult tickets sold

b = number of concession tickets sold

$a + c = 400$ and $15a + 9.5c = 5422.5$

b $\begin{bmatrix} 1 & 1 \\ 15.00 & 9.50 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 400 \\ 5422.50 \end{bmatrix}$

c 295

24 a $J_1 = \begin{bmatrix} 175.5 \\ 260.9 \end{bmatrix}$

b 127 Year 11 jackets and 233 Year 12 jackets

25 a $\begin{bmatrix} \frac{-4}{19} & \frac{-1}{19} & \frac{13}{19} \\ \frac{-7}{19} & \frac{3}{19} & \frac{-1}{19} \\ \frac{8}{19} & \frac{2}{19} & \frac{-7}{19} \end{bmatrix}$

b $\begin{bmatrix} 0 & \frac{7}{3} & \frac{-8}{3} & \frac{10}{3} \\ \frac{1}{3} & \frac{-7}{9} & \frac{11}{9} & \frac{-13}{9} \\ 0 & -1 & 1 & -1 \\ \frac{-1}{3} & \frac{22}{9} & \frac{-23}{9} & \frac{28}{9} \end{bmatrix}$

26 $a = 2, b = -1, c = 3, d = -2$