6

Sequences

6.1 Kick off with CAS

6.2 Arithmetic sequences

6.3 Geometric sequences

6.4 Recurrence relations

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6.1 Kick off with CAS

Exploring the Fibonacci sequence with CAS

The Fibonacci sequence is a sequence of numbers that starts with 1 and 1, after which every subsequent number is found by adding the two previous numbers. Thus the sequence is:

1, 1, 2, 3, 5, 8, 13, 21, 34, …

This sequence is frequently found in nature. For example, the numbers of petals of many flowers fall within this sequence: a lily has 3 petals, a buttercup has 5 petals and a daisy has 34 petals, just to name a few. Within the head of a sunflower, seeds are produced at the centre and then migrate to the outside in spiral patterns, with the numbers of seeds in the spirals being numbers from the Fibonacci sequence.

1 Using CAS and a list and spreadsheet application, generate the first 30 terms of the Fibonacci sequence.

2 If the first 3 numbers in the Fibonacci sequence are called $t_1 = 1$ (term 1), $t_2 = 1$ (term 2) and $t_3 = 2$ (term 3), what is the value of t_{20}?

3 What is the smallest value of n for which $t_n > 1000$?

4 Calculate the ratios of consecutive terms for the first 12 terms; that is,

$$\frac{1}{1} = 1, \frac{2}{1} = 2, \frac{3}{2} = 1.5, \frac{5}{3} = ?, \frac{8}{5} = ?, \frac{13}{8} = ?, \frac{21}{13} = ?, \frac{34}{21} = ?, \frac{55}{34} = ?, \frac{89}{55} = ?, \frac{?}{?} = ?$$

5 What do you notice about the value of the ratios as the terms increase?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

6.2 Arithmetic sequences

Defining mathematical sequences

A **sequence** is a related set of objects or events that follow each other in a particular order. Sequences can be found in everyday life, with some examples being:

- the opening share price of a particular stock each day
- the daily minimum temperature readings in a particular city
- the lowest petrol prices each day
- the population of humans counted each year.

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When data is collected in the order that the events occur, patterns often emerge. Some patterns can be complicated, whereas others are easy to define.

In mathematics, sequences are always ordered, and the links between different terms of sequences can be identified and expressed using mathematical equations.

You may already be familiar with some mathematical sequences, such as the multiples of whole numbers or the square numbers.

Multiples of 3: 3, 6, 9, 12, …

Multiples of 5: 5, 10, 15, 20, …

Square numbers: 1, 4, 9, 16, …

For each of these patterns there is a link between the numbers in the sequence (known as **terms**) and their position in the sequence (known as the **term number**).

The language of mathematical sequences

In general, mathematical sequences can be displayed as:

$$t_1, t_2, t_3, t_4, t_5, ..., t_n$$

where t_1 is the first term, t_2 is the second term, and so on.

The first term of a mathematical sequence can also be referred to as a. The nth term is referred to as t_n (so $t_1 = a$), and n represents the ordered position of the term in the sequence, for example 1st, 2nd, 3rd, …

Sequences expressed as functions

If we consider the term numbers in a sequence as the inputs of a **function**, then the term values of that sequence are the outputs of that function.

INPUT → OUTPUT

Term number → Function → Term value

If we are able to define a sequence as a function, then we can input term numbers into that function to determine any term value in the sequence.

WORKED EXAMPLE 1

Determine the first five terms of the sequence $t_n = 2n + 3$.

THINK	WRITE
1 Substitute $n = 1$ into the function.	$t_1 = 2 \times 1 + 3$ $= 5$
2 Substitute $n = 2$ into the function.	$t_2 = 2 \times 2 + 3$ $= 7$
3 Substitute $n = 3$ into the function.	$t_3 = 2 \times 3 + 3$ $= 9$
4 Substitute $n = 4$ into the function.	$t_4 = 2 \times 4 + 3$ $= 11$
5 Substitute $n = 5$ into the function.	$t_5 = 2 \times 5 + 3$ $= 13$
6 State the answer	The first five terms of the sequence are 5, 7, 9, 11 and 13.

Note: You can see that the terms of the sequence increase by the coefficient of n (i.e. the number n is multiplied by).

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AOS 3
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Arithmetic sequences
Concept summary
Practice questions

Arithmetic sequences

An **arithmetic sequence** is a sequence in which the difference between any two successive terms in the sequence is the same. In an arithmetic sequence, the next term in the sequence can be found by adding or subtracting a fixed value.

First consider the sequence 5, 9, 13, 17, 21. This is an arithmetic sequence, as each term is obtained by adding 4 (a fixed value) to the preceding term.

Now consider the sequence 1, 3, 6, 10, 15. This is not an arithmetic sequence, as each term does not increase by the same constant value.

The common difference

The difference between two consecutive terms in an arithmetic sequence is known as the common difference. If the common difference is positive, the sequence is increasing. If the common difference is negative, the sequence is decreasing.

In an arithmetic sequence, the first term is referred to as *a* and the common difference is referred to as *d*.

WORKED EXAMPLE 2

Determine which of the following sequences are arithmetic sequences, and for those sequences which are arithmetic, state the values of *a* and *d*.

a 2, 5, 8, 11, 14, …

b 4, −1, −6, −11, −16, …

c 3, 5, 9, 17, 33, …

THINK	WRITE
a 1 Calculate the difference between consecutive terms of the sequence.	**a** $t_2 - t_1 = 5 - 2$ $= 3$ $t_3 - t_2 = 8 - 5$ $= 3$ $t_4 - t_3 = 11 - 8$ $= 3$ $t_5 - t_4 = 14 - 11$ $= 3$
2 If the differences between consecutive terms are constant, then the sequence is arithmetic. The first term of the sequence is a and the common difference is d.	The common differences are constant, so the sequence is arithmetic. $a = 2$ and $d = 3$
b 1 Calculate the difference between consecutive terms of the sequence.	**b** $t_2 - t_1 = -1 - 4$ $= -5$ $t_3 - t_2 = -6 - -1$ $= -6 + 1$ $= -5$ $t_4 - t_3 = -11 - -6$ $= -11 + 6$ $= -5$ $t_5 - t_4 = -16 - -11$ $= -16 + 11$ $= -5$
2 If the differences between consecutive terms are constant, then the sequence is arithmetic. The first term of the sequence is a and the common difference is d.	The common differences are constant, so the sequence is arithmetic. $a = 4$ and $d = -5$
c 1 Calculate the difference between consecutive terms of the sequence.	**c** $t_2 - t_1 = 5 - 3$ $= 2$ $t_3 - t_2 = 9 - 5$ $= 4$ $t_4 - t_3 = 17 - 9$ $= 8$ $t_5 - t_4 = 33 - 17$ $= 16$
2 If the differences between consecutive terms are constant, then the sequence is arithmetic.	The common differences are not constant, so the sequence is not arithmetic.

Equations representing arithmetic sequences

If we want to determine any term of an arithmetic sequence, we need to set up an equation to represent the sequence.

Any arithmetic sequence can be expressed by the equation $t_n = a + (n - 1)d$, where t_n is the nth term, a is the first term and d is the common difference.

Therefore, if we know or can determine the values of a and d, we can construct the equation for the sequence.

WORKED EXAMPLE 3

Determine the equations that represent the following arithmetic sequences.

a **3, 6, 9, 12, 15, …**

b **40, 33, 26, 19, 12, …**

THINK	WRITE
a 1 Determine the values of a and d.	**a** $a = 3$ $d = t_2 - t_1$ $= 6 - 3$ $= 3$
2 Substitute the values for a and d into the formula for arithmetic sequences.	$t_n = a + (n - 1)d$ $= 3 + (n - 1) \times 3$ $= 3 + 3(n - 1)$ $= 3 + 3n - 3$ $= 3n$
b 1 Determine the values of a and d.	**b** $a = 40$ $d = t_2 - t_1$ $= 33 - 40$ $= -7$
2 Substitute the values for a and d into the formula for arithmetic sequences.	$t_n = a + (n - 1)d$ $= 40 + (n - 1) \times -7$ $= 40 - 7(n - 1)$ $= 40 - 7n + 7$ $= 47 - 7n$

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Interactivity
Terms of an arithmetic sequence
int-6261

Determining future terms of an arithmetic sequence

After an equation has been set up to represent an arithmetic sequence, we can use this equation to determine any term in the sequence. Simply substitute the value of n into the equation to determine the value of that term.

Determining other values of an arithmetic sequence

We can obtain the values a, d and n for an arithmetic sequence by transposing the equation.

$$a = t_n - (n - 1)d$$

$$d = \frac{t_n - a}{n - 1}$$

$$n = \frac{t_n - a}{d} + 1$$

WORKED EXAMPLE 4

a **Find the 15th term of the sequence 2, 8, 14, 20, 26, …**

b **Find the first term of the arithmetic sequence in which $t_{22} = 1008$ and $d = -8$.**

c Find the common difference of the arithmetic sequence which has a first term of 12 and an 11th term of 102.

d An arithmetic sequence has a first term of 40 and a common difference of 12. Which term number has a value of 196?

THINK	WRITE
a 1 As it has a common difference, this is an arithmetic sequence. State the known values.	**a** $a = 2,\ d = 6,\ n = 15$
2 Substitute the known values into the equation for an arithmetic sequence and solve.	$t_n = a + (n-1)d$ $t_{15} = 2 + (15-1)6$ $= 2 + 14 \times 6$ $= 2 + 84$ $= 86$
3 State the answer	The 15th term of the sequence is 86.
b 1 State the known values of the arithmetic sequence.	**b** $d = -8,\ n = 22,\ t_{22} = 1008$
2 Substitute the known values into the equation to determine the first term and solve.	$a = t_n - (n-1)d$ $= 1008 - (22-1)(-8)$ $= 1008 - (21)(-8)$ $= 1008 - -168$ $= 1008 + 168$ $= 1176$
3 State the answer	The first term of the sequence is 1176.
c 1 State the known values of the arithmetic sequence.	**c** $a = 12,\ n = 11,\ t_{11} = 102$
2 Substitute the known values into the equation to determine the common difference and solve.	$d = \frac{t_n - a}{n-1}$ $= \frac{102 - 12}{11 - 1}$ $= \frac{90}{10}$ $= 9$
3 State the answer	The common difference is 9.
d 1 State the known values of the arithmetic sequence.	**d** $a = 40,\ d = 12,\ t_n = 196$
2 Substitute the known values into the equation to determine the term number and solve.	$n = \frac{t_n - a}{d} + 1$ $= \frac{196 - 40}{12} + 1$ $= 14$
3 State the answer	The 14th term in the sequence has a value of 196.

Graphical displays of sequences

Tables of values

When we draw a graph of a mathematical sequence, it helps to first draw a table of values for the sequence. The top row of the table displays the term number of the sequence, and the bottom of the table displays the term value.

Term number	1	2	3	…	n
Term value					

The data from the table of values can then be used to identify the points to plot in the graph of the sequence.

Drawing graphs of sequences

When we draw a graph of a numerical sequence, the term number is the independent variable, so it appears on the x-axis of the graph. The term value is the dependent value, so it appears on the y-axis of the graph.

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Interactivity
Arithmetic sequences
int-6258

Graphical displays of arithmetic sequences

Because there is a common difference between the terms of an arithmetic sequence, the relationship between the terms is a linear relationship. This means that when we graph the terms of an arithmetic sequence, we can join the points to form a straight line.

When we draw a graph of an arithmetic sequence, we can extend the straight line to determine values of terms in the sequence that haven't yet been determined.

WORKED EXAMPLE 5

An arithmetic sequence is given by the equation $t_n = 7 + 2(n - 1)$.

a Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.

b Plot the graph of the sequence.

c Use your graph of the sequence to determine the 12th term of the sequence.

THINK

a 1 Set up a table with the term number in the top row and the term value in the bottom row.

2 Substitute the first 5 values of n into the equation to determine the missing values.

WRITE/DRAW

a

Term number	1	2	3	4	5
Term value					

$$\begin{aligned} t_1 &= 7 + 2(1 - 1) \\ &= 7 + 2 \times 0 \\ &= 7 + 0 \\ &= 7 \\ t_2 &= 7 + 2(2 - 1) \\ &= 7 + 2 \times 1 \\ &= 7 + 2 \\ &= 9 \\ t_3 &= 7 + 2(3 - 1) \\ &= 7 + 2 \times 2 \\ &= 7 + 4 \\ &= 11 \end{aligned}$$

$$t_4 = 7 + 2(4 - 1)$$
$$= 7 + 2 \times 3$$
$$= 7 + 6$$
$$= 13$$
$$t_5 = 7 + 2(5 - 1)$$
$$= 7 + 2 \times 4$$
$$= 7 + 8$$
$$= 15$$

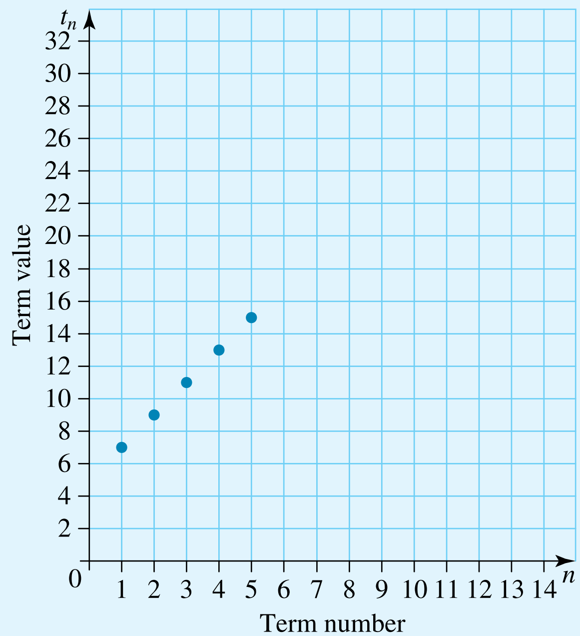
3 Complete the table with the calculated values.

Term number	1	2	3	4	5
Term value	7	9	11	13	15

b **1** Use the table of values to identify the points to be plotted.

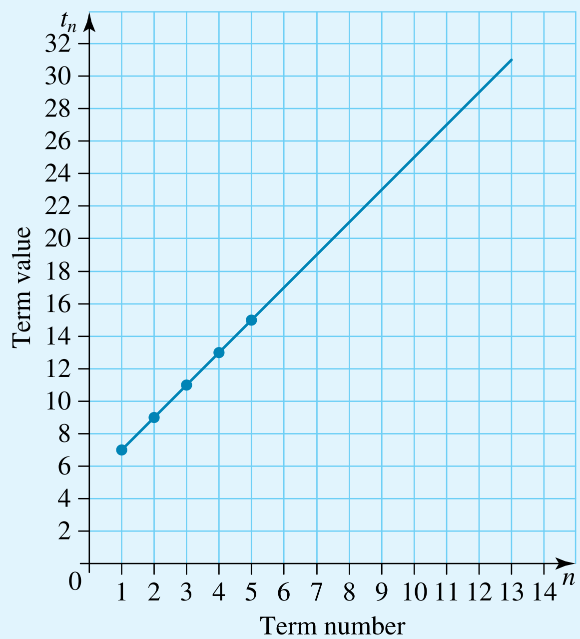
b The points to be plotted are (1, 7), (2, 9), (3, 11), (4, 13) and (5, 15).

2 Plot the points on the graph.

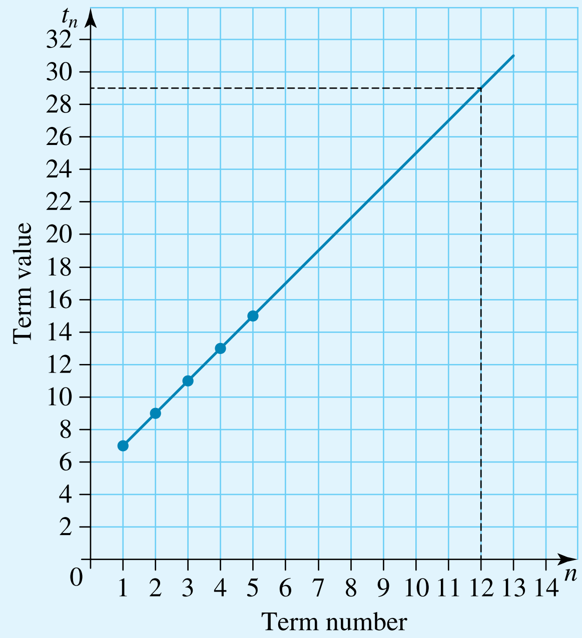


c **1** Join the points with a straight line and extend the line to cover future values of the sequence.

c



2 Read the required value from the graph (when $n = 12$).



3 Write the answer.

The 12th term of the sequence is 29.

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Units 1 & 2
AOS 3
Topic 3
Concept 4

Modelling using arithmetic sequences
Concept summary
Practice questions

Using arithmetic sequences to model practical situations

If we have a practical situation involving linear growth or decay in discrete steps, this situation can be modelled by an arithmetic sequence.

Simple interest

As covered in Topic 3, simple interest is calculated on the original amount of money invested. It is a fixed amount of interest paid at equal intervals, and as such it can be modelled by an arithmetic sequence.

Remember that simple interest is calculated by using the formula $I = \frac{PrT}{100}$, where I is the amount of simple interest, P is the principle, r is the percentage rate and T is the amount of periods.

WORKED EXAMPLE 6

Jelena puts $1000 into an investment that earns simple interest at a rate of 0.5% per month.

a Set up an equation that represents Jelena's situation as an arithmetic sequence, where t_n is the amount in Jelena's account after n months.

b Use your equation from part a to determine the amount in Jelena's account at the end of each of the first 6 months.

c Calculate the amount in Jelena's account at the end of 18 months.

THINK	WRITE
a 1 Use the simple interest formula to determine the amount of simple interest Jelena earns in one month.	**a** $I = \frac{PrT}{100}$ $= \frac{1000 \times 0.5 \times 1}{100}$ $= \frac{500}{100}$ $= 5$
2 Calculate the amount in the account after the first month.	$a = 1000 + 5$ $= 1005$
3 State the known values in the arithmetic sequence equation.	$a = 1005,\ d = 5$
4 Substitute these values into the arithmetic sequence equation.	$t_n = 1005 + 5(n - 1)$
b 1 Use the equation from part **a** to find the values of t_2, t_3, t_4, t_5 and t_6.	**b** $t_2 = 1005 + 5(2 - 1)$ $= 1005 + 5 \times 1$ $= 1005 + 5$ $= 1010$ $t_3 = 1005 + 5(3 - 1)$ $= 1005 + 2 \times 1$ $= 1005 + 10$ $= 1015$ $t_4 = 1005 + 5(4 - 1)$ $= 1005 + 5 \times 3$ $= 1005 + 15$ $= 1020$ $t_5 = 1005 + 5(5 - 1)$ $= 1005 + 5 \times 4$ $= 1005 + 20$ $= 1025$ $t_6 = 1005 + 5(6 - 1)$ $= 1005 + 5 \times 5$ $= 1005 + 25$ $= 1030$
2 Write the answer.	The amounts in Jelena's account at the end of each of the first 6 months are \$1005, \$1010, \$1015, \$1020, \$1025 and \$1030.
c 1 Use the equation from part **a** to find the values of t_{18}.	**c** $t_{18} = 1005 + 5(18 - 1)$ $= 1005 + 5 \times 17$ $= 1005 + 85$ $= 1090$
2 Write the answer.	After 18 months Jelena has \$1090 in her account.

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Units 1 & 2
AOS 3
Topic 3
Concept 7

Depreciation
Concept summary
Practice questions

Depreciating assets

Many items, such as automobiles or electronic equipment, decrease in value over time as a result of wear and tear. At tax time individuals and companies use depreciation of their assets to offset expenses and to reduce the amount of tax they have to pay.

Unit cost depreciation

Unit cost depreciation is a way of depreciating an asset according to its use. For example, you can depreciate the value of a car based on how many kilometres it has driven. The unit cost is the amount of depreciation per unit of use, which would be 1 kilometre of use in the example of the car.

Future value and write-off value

When depreciating the values of assets, companies will often need to know the **future value** of an item. This is the value of that item at that specific time.

The **write-off value** or scrap value of an asset is the point at which the asset is effectively worthless (i.e. has a value of $0) due to depreciation.

WORKED EXAMPLE 7

Loni purchases a new car for $25 000 and decides to depreciate it at a rate of $0.20 per km.

a Set up an equation to determine the value of the car after *n* km of use.

b Use your equation from part a to determine the future value of the car after it has 7500 km on its clock.

THINK	WRITE
a 1 Calculate the value of the car after 1 km of use.	**a** $a = 25\,000 - 0.2$ $= 24\,999.8$
2 State the known values in the arithmetic sequence equation.	$a = 24\,999.8,\ d = -0.2$
3 Substitute these values into the arithmetic sequence equation.	$t_n = a + (n-1)d$ $= 24\,999.8 + (n-1) \times -0.2$ $= 24\,999.8 - 0.2(n-1)$
b 1 Substitute $n = 7500$ into the equation determined in part a.	**b** $t_n = 24\,999.8 - 0.2(n-1)$ $t_{7500} = 24\,999.8 - 0.2(7500-1)$ $= 24\,999.8 - 0.2 \times 7499$ $= 24\,999.8 - 1499.8$ $= 23\,000$
2 Write the answer.	After 7500 km the car will be worth $23 000.

EXERCISE 6.2 Arithmetic sequences

PRACTISE

1 WE1 Determine the first five terms of the sequence $t_n = 5n + 7$.

2 Determine the first five terms of the sequence $t_n = 3n - 5$.

3 WE2 Determine which of the following sequences are arithmetic sequences, and for those sequences which are arithmetic, state the values of a and d.

a 23, 68, 113, 158, 203, …

b 3, 8, 23, 68, 203, …

c $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \ldots$

4 Find the missing values in the following arithmetic sequences.

a 13, −12, −37, f, −87, …

b 2.5, j, 8.9, 12.1, k, …

c $p, q, r, \frac{9}{2}, \frac{25}{4}, \ldots$

5 WE3 Determine the equations that represent the following arithmetic sequences.

a −1, 3, 7, 11, 15, …

b 1.5, −2, −5.5, −8, −11.5

c $\frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \frac{23}{2}, \ldots$

6 Determine the first five terms of the following arithmetic sequences.

a $t_n = 5 + 3(n - 1)$

b $t_n = -1 - 7(n - 1)$

c $t_n = \frac{1}{3} + \frac{2}{3}(n - 1)$

7 WE4 a Find the 20th term of the sequence 85, 72, 59, 46, 33, …

b Find the first value of the arithmetic sequence in which $t_{70} = 500$ and $d = -43$.

8 a Find the common difference of the arithmetic sequence that has a first term of −32 and an 8th term of 304.

b An arithmetic sequence has a first term of 5 and a common difference of 40. Which term number has a value of 85?

c An arithmetic sequence has a first term of 40 and a common difference of 12. Which term number has a value of 196?

9 WE5 An arithmetic sequence is given by the equation $t_n = 5 + 10(n - 1)$.

a Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.

b Plot the graph of the sequence.

c Use your graph of the sequence to determine the 9th term of the sequence.

10 An arithmetic sequence is defined by the equation $t_n = 6.4 + 1.6(n - 1)$.

a Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.

b Plot the graph of the sequence.

c Use your graph of the sequence to determine the 13th term of the sequence.

11 WE6 Grigor puts $1500 into an investment account that earns simple interest at a rate of 4.8% per year.

a Set up an equation that represents Grigor's situation as an arithmetic sequence, where t_n is the amount in Grigor's account after n months.

b Use your equation from part a to determine the amount in Grigor's account after each of the first 6 months.

c Calculate the amount in Grigor's account at the end of 18 months.

12 Justine sets up an equation to model the amount of her money in a simple interest investment account after n months. Her equation is $t_n = 8050 + 50(n - 1)$, where t_n is the amount in Justine's account after n months.

a How much did Justine invest in the account?

b What is the annual interest rate of the investment?

13 WE7 Phillipe purchases a new car for \$24 000 and decides to depreciate it at a rate of \$0.25 per km.

a Set up an equation to determine the value of the car after n km of use.
b Use your equation from part a to determine the future value of the car after it has 12 000 km on its clock.

14 Dougie is in charge of the equipment for his office. He decides to depreciate the value of a photocopier at the rate of x cents for every n copies made. Dougie's equation for the value of the photocopier after n copies is $t_n = 5399.999 - 0.001(n - 1)$

a How much did the photocopier cost?
b What is the rate of depreciation per copy made?

CONSOLIDATE

15 a Find the 15th term of the arithmetic sequence 6, 13, 20, 27, 34, …
b Find the 20th term of the arithmetic sequence 9, 23, 37, 51, 65, …
c Find the 30th term of the arithmetic sequence 56, 48, 40, 32, 24, …
d Find the 55th term of the arithmetic sequence $\frac{72}{5}, \frac{551}{40}, \frac{263}{20}, \frac{501}{40}, \frac{119}{10}, \ldots$

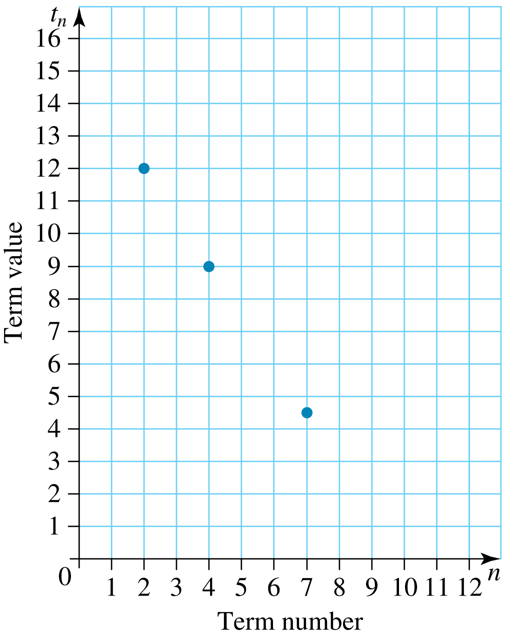
16 a Find the first value of the arithmetic sequence which has a common difference of 6 and a 31st term of 904.
b Find the first value of the arithmetic sequence which has a common difference of $\frac{2}{5}$ and a 40th term of −37.2.
c Find the common difference of an arithmetic sequence which has a first value of 564 and a 51st term of 54.
d Find the common difference of an arithmetic sequence which has a first value of −87 and a 61st term of 43.

17 a An arithmetic sequence has a first value of 120 and a common difference of 16. Which term has a value of 712?
b An arithmetic sequence has a first value of 320 and a common difference of 4. Which term has a value of 1160?

18 Three consecutive terms of an arithmetic sequence are $x - 5$, $x + 4$ and $2x - 7$. Find the value of x.

19 The graph shows some points of an arithmetic sequence.

a What is the common difference between consecutive terms?
b What is the value of the first term of the sequence?
c What is the value of the 12th term of the sequence?



20 Sketch the graph of $t_n = a + (n - 1)d$, where $a = 15$ and $d = 25$, for the first 10 terms.

21 An employee starts a new job with a $60 000 salary in the first year and the promise of a pay rise of $2500 a year.

a How much will her salary be in their 6th year?
b How long will it take for her salary to reach $85 000?

22 Nadia wants to invest her money and decided to place $90 000 into a credit union account earning simple interest at a rate of 6% per year.

a How much interest will Nadia receive after one year?
b What is the total amount Nadia has in the credit union after n years?
c For how long should Nadia keep her money invested if she wants a total of $154 800 returned?

23 Tom bought a car for $23 000, knowing it would depreciate in value by $210 per month.

a What is the value of the car after 18 months?
b By how much does the value of the car depreciate in 3 years?
c How many months will it take for the car to be valued at $6200?

24 A confectionary manufacturer introduces a new sweet and produces 50 000 packets of the sweets in the first week. The stores sell them quickly, and in the following week there is demand for 30% more. In each subsequent week the increase in production is 30% of the original production.

a How many packs are manufactured in the 20th week?
b In which week will the confectionary manufacturer produce 5 540 000 packs?

MASTER

25 A canning machine was purchased for a total of $250 000 and is expected to produce 500 000 000 cans before it is written off.

a By how much does the canning machine depreciate with each can made?
b If the canning machine were to make 40 200 000 cans each year, when will the machine theoretically be written off?
c When will the machine have a book value of $89 200?

26 The local rugby club wants to increase its membership. In the first year they had 5000 members, and so far they have managed to increase their membership by 1200 members per year.

a If the increase in membership continues at the current rate, how many members will they have in 15 years' time?

Tickets for membership in the first year were $200, and each year the price has risen by a constant amount, with memberships in the 6th year costing $320.

b How much would the tickets cost in 15 years' time?

c What is the total membership income in both the first and 15th years?

6.3 Geometric sequences

Geometric sequences

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Units 1 & 2
AOS 3
Topic 3
Concept 5

Geometric sequences
Concept summary
Practice questions

A **geometric sequence** is a pattern of numbers whose consecutive terms increase or decrease in the same ratio.

First consider the sequence 1, 3, 9, 27, 81, … This is a geometric sequence, as each term is obtained by multiplying the preceding term by 3.

Now consider the sequence 1, 3, 6, 10, 15, … This is not a geometric sequence, as the consecutive terms are not increasing in the same ratio.

Common ratios

The ratio between two consecutive terms in a geometric sequence is known as the **common ratio**.

In a geometric sequence, the first term is referred to as *a* and the common ratio is referred to as *r*.

WORKED EXAMPLE 8

Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of a and r.

a 20, 40, 80, 160, 320, …

b $8, 4, 2, 1, \frac{1}{2}, \ldots$

c 3, –9, 27, –81, …

d 2, 4, 6, 8, 10, …

THINK	WRITE
a 1 Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.	**a** $\frac{t_2}{t_1} = \frac{40}{20}$ $= 2$ $\frac{t_3}{t_2} = \frac{80}{40}$ $= 2$ $\frac{t_4}{t_3} = \frac{160}{80}$ $= 2$ $\frac{t_5}{t_4} = \frac{320}{160}$ $= 2$
2 If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is a and the common difference is r.	The ratios between consecutive terms are all 2, so this is a geometric sequence. $a = 20$, $r = 2$

b **1** Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

b $\frac{t_2}{t_1} = \frac{4}{8}$

$= \frac{1}{2}$

$\frac{t_3}{t_2} = \frac{2}{4}$

$= \frac{1}{2}$

$\frac{t_4}{t_3} = \frac{1}{2}$

$\frac{t_5}{t_4} = \frac{\left(\frac{1}{2}\right)}{1}$

$= \frac{1}{2}$

2 If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is a and the common difference is r.

The ratios between consecutive terms are all $\frac{1}{2}$ so this is a geometric sequence.

$a = 8, r = \frac{1}{2}$

c **1** Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

c $\frac{t_2}{t_1} = \frac{-9}{3}$

$= -3$

$\frac{t_3}{t_2} = \frac{27}{-9}$

$= -3$

$\frac{t_4}{t_3} = \frac{-81}{27}$

$= -3$

2 If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is a and the common difference is r.

The ratios between consecutive terms are all -3, so this is a geometric sequence.

$a = 3, r = -3$

d **1** Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

d $\frac{t_2}{t_1} = \frac{4}{2}$

$= 2$

$\frac{t_3}{t_2} = \frac{6}{4}$

$= \frac{3}{2}$

$\frac{t_4}{t_3} = \frac{8}{6}$

$= \frac{4}{3}$

$\frac{t_5}{t_4} = \frac{10}{8}$

$= \frac{5}{4}$

2 If the ratios between consecutive terms are constant, then the sequence is geometric.	All of the ratios between consecutive terms are different, so this is not a geometric sequence.

Equations representing geometric sequences

Any geometric sequence can be represented by the equation $t_n = ar^{n-1}$, where t_n is the nth term, a is the first term and r is the common ratio.

Therefore, if we know or can determine the values of a and r for a geometric sequence, we can construct the equation for the sequence.

WORKED EXAMPLE 9

Determine the equations that represent the following geometric sequences.

a 7, 28, 112, 448, 1792, …

b $8, -4, 2, -1, \frac{1}{2}, \ldots$

THINK	WRITE
a 1 Determine the values of a and r.	**a** $a = 7$ $r = \frac{t_2}{t_1}$ $= \frac{28}{7}$ $= 4$
2 Substitute the values for a and r into the formula for geometric sequences.	$t_n = ar^{n-1}$ $= 7 \times 4^{n-1}$
b 1 Determine the values of a and r.	**b** $a = 8$ $r = \frac{t_2}{t_1}$ $= \frac{-4}{8}$ $= -\frac{1}{2}$
2 Substitute the values for a and r into the formula for geometric sequences.	$t_n = ar^{n-1}$ $= 8 \times \left(-\frac{1}{2}\right)^{n-1}$

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Interactivity
Terms of a geometric sequence
int-6260

Determining future terms of a geometric sequence

After an equation has been set up to represent a geometric sequence, we can use this equation to determine any term in the sequence. Simply substitute the value of n into the equation to determine the value of that term.

Determining other values of a geometric sequence

We can obtain the values a and r for a geometric sequence by transposing the equation.

$$a = \frac{t_n}{r^{n-1}}$$

$$r = \left(\frac{t_n}{a}\right)^{\frac{1}{n-1}}$$

Note: The value of n can also be determined, but this is beyond the scope of this course.

WORKED EXAMPLE 10

a Find the 20th term of the geometric sequence with $a = 5$ and $r = 2$.

b A geometric sequence has a first term of 3 and a 20th term of 1 572 864. Find the common ratio between consecutive terms of the sequence.

c Find the first term of a geometric series with a common ratio of 2.5 and a 5th term of 117.1875.

THINK	WRITE
a 1 Identify the known values in the question.	**a** $a = 5$ $r = 2$ $n = 20$
2 Substitute these values into the geometric sequence formula and solve to find the missing value.	$t_n = ar^{n-1}$ $t_{20} = 5 \times 2^{20-1}$ $= 5 \times 2^{19}$ $= 2\,621\,440$
3 Write the answer.	The 20th term of the sequence is 2 621 440.
b 1 Identify the known values in the question.	**b** $t_{20} = 4\,194\,304$ $a = 4$ $n = 20$
2 Substitute these values into the formula to calculate the common ratio and solve to find the missing value.	$r = \left(\frac{t_n}{a}\right)^{\frac{1}{n-1}}$ $= \left(\frac{1\,572\,864}{3}\right)^{\frac{1}{20-1}}$ $= 524\,288^{\frac{1}{19}}$ $= 2$
3 Write the answer.	The common ratio between consecutive terms of the sequence is 2.
c 1 Identify the known values in the question.	**c** $t_5 = 117.1875$ $r = 2.5$ $n = 5$

2 Substitute these values into the formula to calculate the first term and solve to find the missing value.

$$a = \frac{t_n}{r^{n-1}}$$
$$= \frac{117.1875}{2.5^{5-1}}$$
$$= \frac{117.1875}{2.5^4}$$
$$= \frac{117.1875}{39.0625}$$
$$= 3$$

3 Write the answer.

The first term of the sequence is 3.

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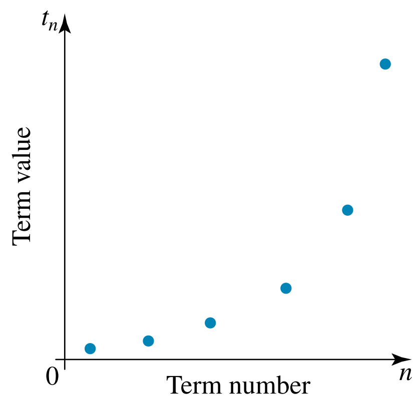
Interactivity
Geometric sequences
int-6259

Graphs of geometric sequences

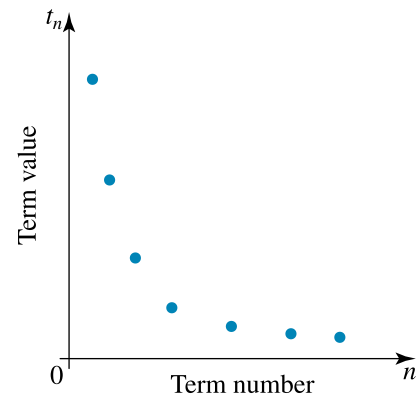
The shape of the graph of a geometric sequence depends on the value of r.

- When $r > 1$, the values of the terms increase or decrease at an exponential rate.
- When $0 < r < 1$, the values of the terms converge towards 0.
- When $-1 < r < 0$, the values of the terms oscillate on either side of 0 but converge towards 0.
- When $r < -1$, the values of the terms oscillate on either side of 0 and move away from the starting value at an exponential rate.

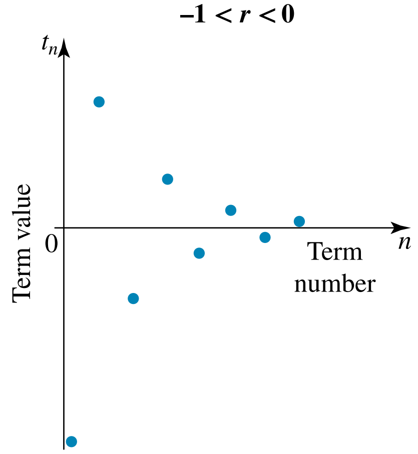
$r > 1$



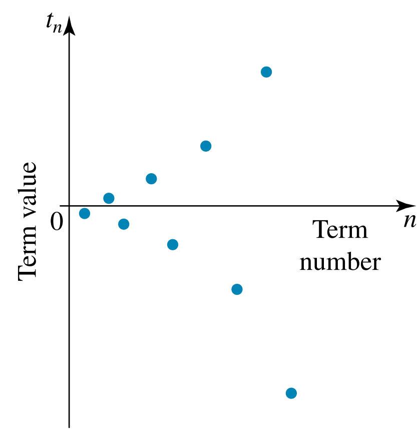
$0 < r < 1$



$-1 < r < 0$



$r < -1$



WORKED EXAMPLE 11

A geometric sequence is defined by the equation $t_n = 5 \times 2^{n-1}$.

a Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.

b Plot the graph of the sequence.

THINK

a 1 Set up a table with the term number in the top row and the term value in the bottom row.

WRITE/DRAW

a

Term number	1	2	3	4	5
Term value					

2 Substitute the first 5 values of n into the equation to determine the missing values.

$$t_1 = 5 \times 2^{1-1} = 5 \times 2^0 = 5 \times 1 = 5$$

$$t_2 = 5 \times 2^{2-1} = 5 \times 2^1 = 5 \times 2 = 10$$

$$t_3 = 5 \times 2^{3-1} = 5 \times 2^2 = 5 \times 4 = 20$$

$$t_4 = 5 \times 2^{4-1} = 5 \times 2^3 = 5 \times 8 = 40$$

$$t_1 = 5 \times 2^{5-1} = 5 \times 2^4 = 5 \times 16 = 80$$

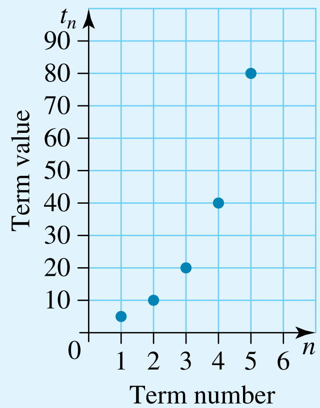
3 Complete the table with the calculated values.

Term number	1	2	3	4	5
Term value	5	10	20	40	80

b 1 Use the table of values to identify the points to be plotted.

b The points to be plotted are (1, 5), (2, 10), (3, 20), (4, 40) and (5, 80).

2 Plot the points on the graph.



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Units 1 & 2

AOS 3

Topic 3

Concept 6

Modelling using geometric sequences

Concept summary

Practice questions

Using geometric sequences to model practical situations

If we have a practical situation involving geometric growth or decay in discrete steps, this situation can be modelled by a geometric sequence.

Compound interest

As covered in Topic 3, compound interest is calculated on the sum of an investment at the start of each compounding period. The amount of interest accrued varies throughout the life of the investment and can be modelled by a geometric sequence.

Remember that simple interest is calculated by using the formula $A = P\left(1 + \frac{r}{100}\right)^n$, where A is the total amount of the investment, P is the principle, r is the percentage rate and n is the number of compounding periods.

WORKED EXAMPLE 12

Alexis puts $2000 into an investment account that earns compound interest at a rate of 0.5% per month.

a **Set up an equation that represents Alexis's situation as a geometric sequence, where t_n is the amount in Alexis' account after n months.**

b **Use your equation from part a to determine the amount in Alexis's account at the end of each of the first 6 months.**

c **Calculate the amount in Alexis's account at the end of 15 months.**

THINK	WRITE
a 1 Determine the amounts in the account after each of the first two months.	**a** $A = P\left(1 + \frac{r}{100}\right)^n$ $= 2000\left(1 + \frac{0.5}{100}\right)^1$ $= 2000 \times 1.005$ $= 2010$ $A = P\left(1 + \frac{r}{100}\right)^n$ $= 2000\left(1 + \frac{0.5}{100}\right)^2$ $= 2000 \times 1.005^2$ $= 2020.05$
2 Calculate the common ratio between consecutive terms.	$r = \frac{t_2}{t_1}$ $= \frac{2020.05}{2010}$ $= 1.005$
3 State the known values in the geometric sequence equation.	$a = 2010,\ r = 1.005$

4 Substitute these values into the geometric sequence equation.	$t_n = 2010 \times 1.005^{n-1}$
b **1** Use the equation from part **a** to find the values of t_3, t_4, t_5 and t_6. Round all values correct to 2 decimal places.	**b** $t_3 = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{3-1}$ $= 2010 \times 1.005^2$ $= 2030.150...$ ≈ 2030.15 $t_4 = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{4-1}$ $= 2010 \times 1.005^3$ $= 2040.301...$ ≈ 2040.30 $t_5 = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{5-1}$ $= 2010 \times 1.005^4$ $= 2050.502...$ ≈ 2050.50 $t_6 = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{6-1}$ $= 2010 \times 1.005^5$ $= 2060.755...$ ≈ 2060.76
2 Write the answer.	The amounts in Alexis' account at the end of each of the first 6 months are \$2010, \$2020.05, \$2030.15, \$2040.30, \$2050.50 and \$2060.76.
c **1** Use the equation from part **a** to find the values of t_{15}, rounding your answer correct to 2 decimal place.	**c** $t_{15} = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{15-1}$ $= 2010 \times 1.005^{14}$ $= 2155.365...$ ≈ 2155.37
2 Write the answer.	After 15 months Alexis has \$2155.37 in her account.

Note: The common ratio in the geometric sequence equation is equal to $1 + \frac{r}{100}$ (from the compound interest formula).

Reducing balance depreciation

Another method of depreciation is **reducing balance depreciation**. When an item is depreciated by this method, rather than the value of the item depreciating by a fixed amount each year, it depreciates by a percentage of the previous future value of the item.

Due to the nature of reducing balance depreciation, we can represent the sequence of the future values of an item that is being depreciated by this method as a geometric sequence.

WORKED EXAMPLE 13

A hot water system purchased for $1250 is depreciated by the reducing balance method at a rate of 8% p.a.

a Set up an equation to determine the value of the hot water system after n years of use.

b Use your equation from part a to determine the future value of the hot water system after 6 years of use (correct to the nearest cent).

THINK	WRITE
a 1 Calculate the common ratio by identifying the value of the item in any given year as a percentage of the value in the previous year. Convert the percentage to a ratio by dividing by 100.	a $100\% - 8\% = 92\%$ Each year the value of the item is 92% of the previous value. $92\% = \frac{92}{100} = 0.92$, $r = 0.92$
2 Calculate the value of the hot water system after 1 year of use.	$a = 1250 \times 0.92 = 1150$
3 Substitute the values of a and r into the geometric sequence equation.	$t_n = 1150 \times 0.92^{n-1}$
b 1 Substitute $n = 6$ into the equation determined in part a. Give your answer correct to 2 decimal places.	b $t_n = 1150 \times 0.92^{n-1}$ $t_6 = 1150 \times 0.92^{6-1} = 1150 \times 0.92^5 = 757.943... \approx 757.94$
2 Write the answer.	After 6 years the book value of the hot water system is $757.94.

EXERCISE 6.3 Geometric sequences

PRACTISE

1 WE8 Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of a and r.

a 3, 6, 12, 24, 48, …

b $\frac{1}{2}, \frac{5}{4}, \frac{25}{8}, \frac{125}{16}, ...$

c 9, 6, 3, 0, –3, …

d $\frac{1}{2}, \frac{1}{5}, \frac{2}{25}, \frac{4}{125}, ...$

2 Find the missing values in the following geometric sequences.

a 1, 6, c, 216, 1296

b 3, g, h, −24, 48

c p, q, s, 300, 1500

3 WE9 Determine the equations that represent the following geometric sequences.

a −1, −5, −25, −125, −625, …

b 7, −3.5, 1.75, –0.875, 0.4375

c $\frac{5}{6}, \frac{5}{9}, \frac{10}{27}, \frac{20}{81}, \frac{40}{243}, ...$

4 Determine the first five terms of the following arithmetic sequences.

a $t_n = -2 \times 3^{n-1}$

b $t_n = 4 \times \left(\frac{1}{3}\right)^{n-1}$

c $t_n = \frac{1}{4} \times \left(-\frac{3}{2}\right)^{n-1}$

5 WE10 a Find the 15th term of the geometric sequence with $a = 4$ and $r = 3$.

b A geometric sequence has a first term of 2 and a 12th term of 97 656 250. Find the common ratio between consecutive terms of the sequence.

c Find the first term of a geometric series with a common ratio of $-\frac{1}{2}$ and a 6th term of 13.125.

6 a Find the 11th term of the geometric sequence with a first value of 1.2 and a common ratio of 4.

b A geometric sequence has a first term of −1.5 and a 10th term of 768. Find the common ratio between consecutive terms of the sequence.

c Find the first term of a geometric series with a common ratio of 0.4 and a 6th term of 6.5536.

7 WE11 A geometric sequence is defined by the equation $t_n = 64 \times \left(\frac{1}{2}\right)^{n-1}$.

a Draw a table of values showing the term number and term value for the first 5 terms of the sequence.

b Plot the graph of the sequence.

8 A geometric sequence is defined by the equation $t_n = 1.5 \times 3^{n-1}$.

a Draw a table of values showing the term number and term value for the first 5 terms of the sequence.

b Plot the graph of the sequence.

9 WE12 Hussein puts $2500 into an investment that earns compound interest at a rate of 0.3% per month.

a Set up an equation that represents Hussein's situation as a geometric sequence, where t_n is the amount in Hussein's account after n months.

b Use your equation from part a to determine the amount in Hussein's account after each of the first 6 months.

c Calculate the amount in Hussein's account at the end of 15 months.

10 Tim sets up an equation to model the amount of his money in a compound interest investment account after n months. His equation is $t_n = 4515.75 \times 1.0035^{n-1}$, where t_n is the amount in his account after n months.

a How much did Tim invest in the account?

b What is the annual interest rate of the investment?

11 WE13 A refrigerator purchased for $1470 is depreciated by the reducing balance method at a rate of 7% p.a.

a Set up an equation to determine the value of the refrigerator after n years of use.

b Use your equation from part a to determine the future value of the refrigerator after 8 years of use.

12 Ivy buys a new oven and decides to depreciate the value of the oven by the reducing balance method.

Ivy's equation for the value of the oven after n years is $t_n = 1665 \times 0.925^{n-1}$.

a How much did the oven cost?

b What is the annual rate of depreciation for the oven?

CONSOLIDATE

13 Which of the following are geometric sequences? Where applicable state the first term and common ratio.

a 3, 15, 75, 375, 1875, …

b 7, 13, 25, 49, 97, …

c –8, 24, –72, 216, –648, …

d 128, 32, 8, 2, $\frac{1}{2}$, ...

e 2, 6, 12, 20, 30, …

f $3, 3\sqrt{3}, 9, 9\sqrt{3}, 27, ...$

14 What is the value of x in the following geometric sequences?

a x, 14, 28, …

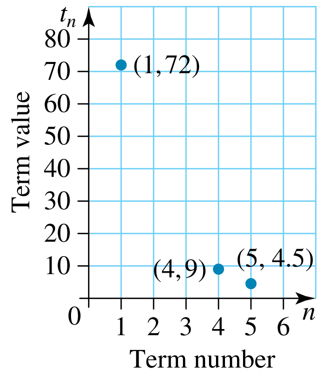
b $2x$, $4x$, $8 + 6x$, …

c $x + 1$, $3x + 3$, $10x + 5$, …

15 a Find the first four terms of the geometric sequence where the 6th term is 243 and the 8th term is 2187.

b Find the first four terms of the geometric sequence where the 3rd term is 331 and the 5th term is 8275.

16 Find the values of the 2nd and 3rd terms of the geometric sequence shown in the following graph.



17 A geometric sequence has a 1st term of 200 and a 6th term of 2.048. Identify the values of the 2nd, 3rd, 4th and 5th terms.

18 The number of ants in a colony doubles every week. If there are 2944 ants in the colony at the end of 8 weeks, how many ants were in the colony at the end of the first week?

19 Jonas starts a new job with a salary of \$55 000 per year and the promise of a 3% pay rise for each subsequent year in the job.

a Write an equation to determine Jonas' salary in his nth year in the job.

b How much will Jonas earn in his 5th year in the job?

20 The 1st term of a geometric sequence is 13 and the 3rd term of the same sequence is 117.

a Explain why there are two possible values for the common ratio of the sequence.

b Calculate both possible values of the 6th term of the sequence.

21 Julio's parents invest \$5000 into a college fund on his 5th birthday. The fund pays a compound interest rate of 5.5% p.a. How much will the fund be worth when Julio turns 18?

22 A meteoroid is burning up as it passes through the Earth's atmosphere. For every 5 km it travels, the mass of the meteoroid decreases by 5%. At the start of its

descent into the Earth's atmosphere, at 100 km above ground level, the mass of the meteoroid is 675 g.

a Formulate an equation to determine the mass of the meteoroid after each 5-km increment of its descent.

b What is the mass of the meteoroid when it hits the Earth, correct to 2 decimal places?

MASTER

23 The number of pieces of stone used to build a pyramid decreases in a ratio of $\frac{1}{3}$ for each layer of the pyramid. The pyramid has 9 layers. The top (9th) layer of the pyramid needed only 2 stones.

a How many stones were needed for the base layer of the pyramid?

b Write an equation to express how many stones were needed for the nth layer of the pyramid.

c How many stones were needed for the entire pyramid?

24 The populations of Melbourne and Sydney are projected to grow steadily over the next 20 years. A government agency predicts that the population of Melbourne will grow at a steady rate of 2.6% per year and the population of Sydney will grow at a steady rate of 1.7% per year.

a If the current population of Melbourne is 4.35 million, formulate an equation to estimate the population of Melbourne after n years.

b If the current population of Sydney is 4.65 million, formulate an equation to estimate the population of Sydney after n years.

c Using CAS, determine how long it will take for the population of Melbourne to exceed the population of Sydney. Give your answer correct to the nearest year.

6.4 Recurrence relations

Using first-order linear recurrence relations to generate number sequences

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Units 1 & 2
AOS 3
Topic 3
Concept 2

Linear recurrence relations
Concept summary
Practice questions

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Interactivity
Initial values and first-order recurrence relations
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In a recurrence relation, the terms of a sequence are dependent on the previous terms of the sequence. A first-order linear recurrence relation is a relation whereby the terms of the sequence depend only on the previous term of the sequence, which means that we need only an initial value to be able to generate all remaining terms of the sequence.

In a recurrence relation, the nth term is represented by t_n, with the term directly after t_n being represented by t_{n+1} and the term directly before t_n being represented by t_{n-1}. The initial value of the sequence is represented by the term t_1.

If the initial value in a recurrence relation changes, then the whole sequence changes. If we are not given an initial value, we cannot determine any terms in the sequence.

WORKED EXAMPLE 14 Determine the first five terms of the sequence represented by the recurrence relation $t_n = 2t_{n-1} + 5$, given that $t_1 = 8$.

THINK	WRITE
1 Identify what the recurrence relation means.	$t_n = 2t_{n-1} + 5$ Each term of the sequence is given by multiplying the previous term of the sequence by 2 and adding 5 to the result.
2 Use the recurrence relation to determine the value of t_2.	$t_2 = 2t_1 + 5$ $= 2 \times 8 + 5$ $= 16 + 5$ $= 21$
3 Use the recurrence relation to determine the value of t_3.	$t_3 = 2t_2 + 5$ $= 2 \times 21 + 5$ $= 42 + 5$ $= 47$
4 Use the recurrence relation to determine the value of t_4.	$t_4 = 2t_3 + 5$ $= 2 \times 47 + 5$ $= 94 + 5$ $= 99$
5 Use the recurrence relation to determine the value of t_5.	$t_5 = 2t_4 + 5$ $= 2 \times 99 + 5$ $= 198 + 5$ $= 203$
6 Write the answer.	The first five terms of the sequence are 8, 21, 47, 99 and 203.

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Interactivity
First-order recurrence relations with a common difference
int-6264

Using a recurrence relation to generate arithmetic sequences

If we know the values of a and d in an arithmetic sequence, we can set up a recurrence relation to generate the sequence.

A recurrence relation representing an arithmetic sequence will be of the form $t_{n+1} = t_n + d$, $t_1 = a$.

WORKED EXAMPLE 15 Set up a recurrence relation to represent the arithmetic sequence −9, −5, −1, 3, 7, …

THINK	WRITE
1 Determine the common difference by subtracting the first term from the second term.	$d = -5 - -9$ $= -5 + 9$ $= 4$
2 t_1 represents the first term of the sequence.	$t_1 = -9$
3 Set up the recurrence relation with the given information.	$t_{n+1} = t_n + 4,\ t_1 = -9$

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Interactivity
First-order recurrence relations with a common ratio
int-6263

Using a recurrence relation to generate geometric sequences

If we know the values of a and r in a geometric sequence, we can set up a recurrence relation to generate the sequence.

A recurrence relation representing a geometric sequence will be of the form $t_{n+1} = rt_n$, $t_1 = a$.

WORKED EXAMPLE 16

Set up a recurrence relation to represent the sequence 10, 5, 2.5, 1.25, 0.625, …

THINK	WRITE
1 Determine the common ratio by dividing the second term by the first term.	$r = \frac{t_2}{t_1} = \frac{5}{10} = \frac{1}{2}$
2 t_1 represents the first term of the sequence.	$t_1 = 10$
3 Set up the recurrence relation with the given information.	$t_{n+1} = \frac{1}{2}t_n,\ t_1 = 10$

Using recurrence relations to model practical situations

When there is a situation that can be modelled by an arithmetic or geometric sequence, we can use a recurrence relation to model it. The first step we need to take is to decide whether the information suggests an arithmetic or geometric sequence. If there is a common difference between terms, we can use an arithmetic sequence, and if there is a common ratio between terms, we can use a geometric sequence.

Spotting arithmetic sequences

Arithmetic sequences are sequences that involve linear growth or decay. Examples include simple interest loans or investments, the revenue from the sale of a certain amount of items of the same price, and the number of flowers left in a field if the same amount is harvested each day.

Spotting geometric sequences

Geometric sequences are sequences that involve geometric growth or decay. Examples include compound interest loans or investments, the reducing height of a bouncing ball, and the number of bacteria in a culture after x periods of time.

If percentages are involved in generating a sequence of numbers, this can result in a geometric sequence.

When there is a percentage increase of x percent between terms, the value of the common ratio, r, will be $\left(1 + \frac{x}{100}\right)$.
Similarly, when there is a percentage decrease of x percent between terms, the value of the common ratio, r, will be $\left(1 - \frac{x}{100}\right)$.

WORKED EXAMPLE 17

According to the International Federation of Tennis, a tennis ball must meet certain bounce regulations. The test involves the dropping of a ball vertically from a height of 254 cm and then measuring the rebound height. To meet the regulations, the ball must rebound 135 to 147 cm high, just over half the original distance.

Janine decided to test the ball bounce theory out. She dropped a ball from a height of 200 cm. She found that it bounced back up to 108 cm, with the second rebound reaching 58.32 cm and the third rebound reaching 31.49 cm.

a Set up a recurrence relation to model the bounce height of the ball.

b Use your relation from part **a** to estimate the height of the 4th and 5th rebounds, giving your answers correct to 2 decimal places.

c Sketch the graph of the number of bounces against the height of each bounce.

THINK	WRITE/DRAW
a 1 List the known information.	**a** 1st bounce: 108 cm 2nd bounce: 58.32 cm 3rd bounce: 31.49 cm
2 Check if there is a common ratio between consecutive terms. If so, this situation can be modelled using a geometric sequence.	$\frac{t_2}{t_1} = \frac{58.32}{108}$ $= 0.54$ $\frac{t_3}{t_2} = \frac{31.49}{58.32}$ $= 0.539...$ ≈ 0.54 There is a common ratio between consecutive terms of 0.54.
3 Set up the equation to represent the geometric sequence.	$a = 108$ $r = 0.54$ $t_{n+1} = rt_n, t_1 = a$ $t_{n+1} = 0.54t_n, t_1 = 108$
b 1 Use the formula from part **a** to find the height of the 4th rebound ($n = 4$).	**b** $t_3 = 31.49$ $t_{n+1} = 0.54t_n$ $t_4 = 0.54t_3$ $= 0.54 \times 31.49$ $= 17.0046$ $= 17.00$ (correct to 2 decimal places)
2 Use the formula from part **a** to find the height of the 5th rebound ($n = 5$).	$t_{n+1} = 0.54t_n$ $t_5 = 0.54t_4$ $= 0.54 \times 17.00$ $= 9.18$
3 Write the answer.	The estimated height of the 4th rebound is 17.00 cm, and the estimated height of the 5th rebound is 9.18 cm.

c **1** Draw up a table showing the bounce number against the rebound height.

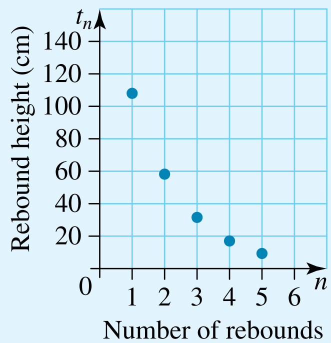
c

Bounce number	1	2	3	4	5
Rebound height (cm)	108	58.32	31.49	17.00	9.18

2 Identify the points to be plotted on the graph.

The points to be plotted are (1, 108), (2, 58.32), (3, 31.49), (4, 17.01) and (5, 9.18).

3 Plot the points on the graph.



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Units 1 & 2
AOS 3
Topic 3
Concept 8

Generation and evaluation of the Fibonacci sequence
Concept summary
Practice questions

The Fibonacci sequence

In 1202, the Italian mathematician Leonardo Fibonacci introduced the Western world to a unique sequence of numbers which we now call the **Fibonacci sequence**.

> **The Fibonacci sequence begins with two 1s, and every subsequent term of the sequence is found by adding the two previous terms, giving the sequence:**
> **1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …**

Generating the Fibonacci sequence using a recurrence relation

Unlike the first-order recurrence relations that we have previously used to represent sequences, the Fibonacci sequence depends on two previous terms, and is therefore a second-order recurrence relation.

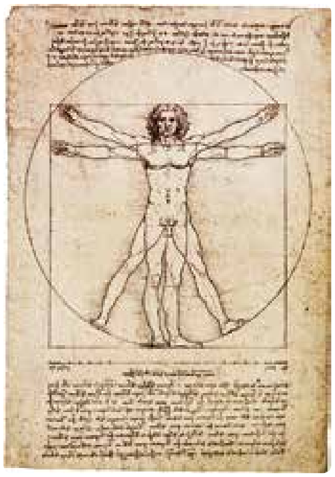
> **The Fibonacci sequence can be represented by the recurrence relation $F_{n+2} = F_n + F_{n+1}$, $F_1 = 1$, $F_2 = 1$.**

The Golden Ratio

The ratios between consecutive terms of the Fibonacci sequence is not a fixed ratio, as with the geometric sequences we've studied. However, the ratios do converge on a number that has special mathematical significance.

Ratio	$\frac{t_2}{t_1} = \frac{1}{1}$	$\frac{t_3}{t_2} = \frac{2}{1}$	$\frac{t_4}{t_3} = \frac{3}{2}$	$\frac{t_5}{t_4} = \frac{5}{3}$	$\frac{t_6}{t_5} = \frac{8}{5}$	$\frac{t_7}{t_6} = \frac{13}{8}$	$\frac{t_8}{t_7} = \frac{21}{13}$	$\frac{t_9}{t_8} = \frac{34}{21}$	$\frac{t_{10}}{t_9} = \frac{55}{34}$
Value	1	2	1.5	1.666…	1.6	1.625	1.615…	1.619…	1.617…

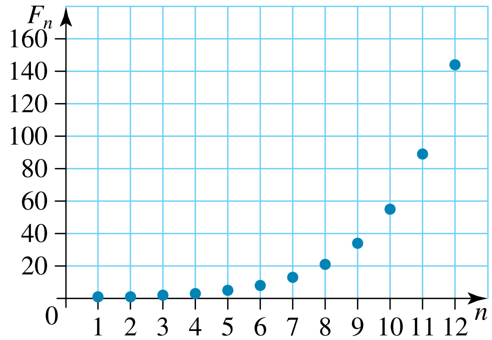
The number that the ratios converge to is called the **Golden Ratio**. It has an exact value of $\frac{1+\sqrt{5}}{2}$.



Throughout history many people have believed that the secrets of beauty lie in the Golden Ratio. Leonardo Da Vinci drew his picture of the Vitruvian Man using the Golden Ratio, and parts of the face of the Mona Lisa are in the proportions of the Golden Ratio.

Graphing the Fibonacci sequence

If you plot the graph of the term numbers of the Fibonacci sequence against the term values, the pattern forms a smooth curve, similar to the graphs of geometric sequences.



Variations of the Fibonacci sequence

The standard Fibonacci sequence begins with two 1s, which are used to formulate the rest of the sequence. If we change these two starting numbers, we get alternative versions of the Fibonacci sequence. For example, if the first two numbers are 2 and 5, the sequence becomes: 2, 5, 7, 12, 19, 31, 50, 81, 131, …

WORKED EXAMPLE 18

State the first 8 terms of the variation of the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_n + F_{n+1}$, $F_1 = -1$, $F_2 = 5$.

THINK	WRITE
1 State the known terms.	$F_1 = -1,\ F_2 = 5$
2 Use the recurrence relation to generate the remaining terms.	$F_{n+2} = F_n + F_{n+1}$ $F_3 = F_1 + F_2$ $= -1 + 5$ $= 4$ $F_4 = F_2 + F_3$ $= 5 + 4$ $= 9$ $F_5 = F_3 + F_4$ $= 4 + 9$ $= 13$ $F_6 = F_4 + F_5$ $= 9 + 13$ $= 22$

$$F_7 = F_5 + F_6$$
$$= 13 + 22$$
$$= 35$$
$$F_8 = F_6 + F_7$$
$$= 22 + 35$$
$$= 57$$

3 Write the answer. The first 8 terms of the sequence are −1, 5, 4, 9, 13, 22, 35, 57.

EXERCISE 6.4 Recurrence relations

PRACTISE

1 **WE14** Determine the first five terms of the sequence represented by the recurrence relation $t_n = 0.5t_{n-1} + 8$, given that $t_1 = 24$.

2 Determine the first five terms of the sequence represented by the recurrence relation $t_n = 3t_{n-1} - 4$, given that $t_1 = 2$.

3 **WE15** Set up a recurrence relation to represent the arithmetic sequence 2, −3, −8, −12, −17.

4 An arithmetic sequence is represented by the recurrence relation $t_{n+1} = t_n + 3.5, t_1 = -2.2$. Determine the first 5 terms of the sequence.

5 **WE16** Set up a recurrence relation to represent the geometric sequence 2.5, −7.5, 22.5, −67.5, 202.5, …

6 A geometric sequence is represented by the recurrence relation $t_{n+1} = -3.5t_n, t_1 = -4$. Determine the first five terms of the sequence.

7 **WE17** Eric decided to test the rebound height of a tennis ball. He dropped a ball from a height of 300 cm and found that it bounced back up to 165 cm, with the second rebound reaching 90.75 cm, and the third rebound reaching 49.91 cm.

a Set up a recurrence relation to model the bounce height of the ball.
b Use your relation from part **a** to estimate the height of the 4th and 5th rebounds, giving your answers correct to 2 decimal places.
c Sketch the graph of the number of bounces against the height of the bounce

8 Rosanna decided to test the ball rebound height of a basketball. She dropped the basketball from a height of 500 cm and noted that each successive rebound was two-fifths of the previous height.

a Set up a recurrence relation to model the bounce height of the ball.
b Use your relation to estimate the heights of the first 5 rebounds, correct to 2 decimal places.
c Sketch the graph of the first 5 bounces against the rebound height.

9 **WE18** State the first 8 terms of the variation of the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_n + F_{n+1}, F_1 = 3, F_2 = -5$.

10 The Lucas sequence is a special variation of the Fibonacci sequence that starts with the numbers 2 and 1. Determine the first 10 numbers of the Lucas sequence.

CONSOLIDATE

11 The 3rd and 4th terms of an arithmetic sequence are −7 and −11.5. Set up a recurrence relation to define the sequence.

12 The 4th and 5th terms of a geometric sequence are −4 and −1. Set up a recurrence relation to define the sequence.

13 A variation of the Fibonacci sequence has a 3rd term of −2 and a 4th term of 6. Determine the recurrence relation for this sequence.

14 Brett invests $18 000 in an account paying simple interest. After 3 months he has $18 189 in his account.

a Set up a recurrence relation to determine the amount in Brett's account after n months.

b How much will Brett have in his account after 7 months?

15 Graph the first 7 terms of the variation of the Fibonacci sequence that starts with the numbers −3 and 3.

16 Cassandra has $6615 in her bank account after 2 years and $6945.75 in her bank account after 3 years. Her account pays compound interest.

a Set up a recurrence relation to determine the amount in Cassandra's account after n years.

b How much does Cassandra have in her account after 5 years?

17 An ice shelf is shrinking at a rate of 1200 km^2 per year. When measurements of the ice shelf began, the area of the shelf was 37 000 km^2.

a Create a recurrence relation to express the area of the ice shelf after n years.

b Use your relation to determine the area of the ice shelf after each of the first 6 years.

c Plot a graph showing the area of the shrinking ice shelf over time.

18 The number of bacteria in a colony is increasing in line with a second-order recurrence relation of the form $t_{n+2} = 2t_n + t_{n+1}$, $t_1 = 3$, $t_2 = 9$, where n is the time in minutes. Determine the amount of bacteria in the colony after each of the first 10 minutes.

19 A bouncing ball rebounds to 70% of its previous height.

a From how high would the ball have to be dropped for the 10th bounce to reach 50 cm in height? Give your answer correct to 1 decimal place.

b Define a recurrence relation to determine the height of the ball after n bounces.

20 An abandoned island is slowly being overrun with rabbits. The population of the rabbits is approximately following a Fibonacci sequence. The estimated number of rabbits after 4 years of monitoring is 35 000, and the estimated number after 5 years of monitoring is 55 000.

a Estimate the number of rabbits after the first year of monitoring.

b Create a recurrence relation to determine the number of rabbits after n years.

c Is it realistic to expect the population of rabbits to continue to increase at this rate? Explain your answer.

MASTER

21 Luke and Lucinda are siblings who are given $3000 to invest by their parents. Luke invests his $3000 in a simple interest bond paying 4.8% p.a., and Lucinda invests her $3000 in a compound interest bond paying 4.3% p.a.

a Write a recurrence relation to express the amount in Luke's account after n years.

b Write a recurrence relation to express the amount in Lucinda's account after n years.

c Determine the amount in each of their accounts for the first 7 years.

d Draw a graph showing the amount in each account over the first 7 years.

22 'Variations of the Fibonacci sequence will always tend towards plus or minus infinity.'

By altering the two starting numbers of the Fibonacci sequence, determine whether this statement is true or not.

6.5 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

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- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
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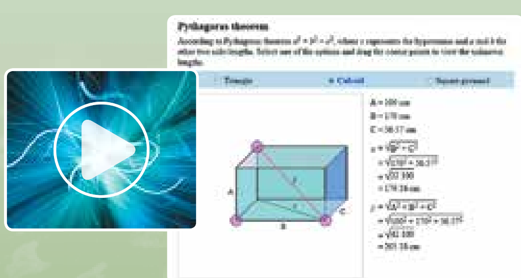

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Units 1 & 2 | **Sequences**

Sit topic test

6 Answers

EXERCISE 6.2

1 12, 17, 22, 27, 32

2 −2, 1, 4, 7, 10

3 a Arithmetic; $a = 23$, $d = 45$

b Not arithmetic

c Arithmetic; $a = \frac{1}{2}$, $d = \frac{1}{4}$

4 a $f = -62$

b $j = 5.7$, $k = 15.3$

c $p = -\frac{3}{4}$, $q = 1$, $r = \frac{11}{4}$

5 a $t_n = -1 + 4(n - 1)$

b $t_n = 1.5 - 3.5(n - 1)$

c $t_n = \frac{7}{2} + 2(n - 1)$

6 a 5, 8, 11, 14, 17

b −1, −8, −15, −22, −29

c $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, 3$

7 a −162 b 3467

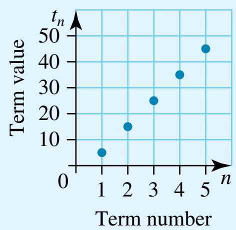
8 a 48 b The 3rd term

c The 14th term

9 a

Term number	1	2	3	4	5
Term value	5	15	25	35	45

b

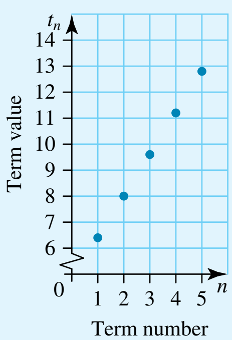


c 85

10 a

Term number	1	2	3	4	5
Term value	6.4	8	9.6	11.2	12.8

b



c 25.6

11 a $t_n = 1506 + 6(n - 1)$

b $1506, $1512, $1518, $1524, $1530, $1536

c $1608

12 a $8000 b 7.5%

13 a $t_n = 23\,999.75 - 0.25(n - 1)$

b $21 000

14 a $5400 b 0.1 cents

15 a 104 b 275 c −176 d $-\frac{387}{20}$

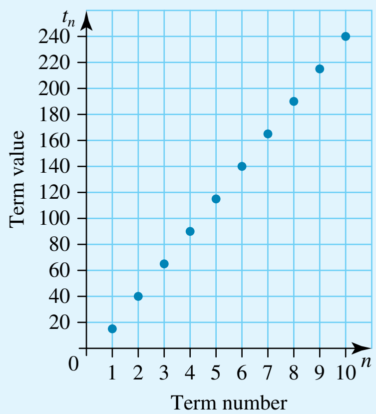
16 a 724 b −52.8 c −10.2 d $\frac{13}{6}$

17 a The 38th term b The 211th term

18 $x = 20$

19 a −1.5 b 13.5 c −3

20



21 a $72 500 b 11 years

22 a $5400

b $t_n = 95\,400 + 5400(n - 1)$

c 12 years

23 a $19 220

b $7560

c 80 months

24 a 335 000

b The 367th week

25 a 0.05 cents

b In the 13th year

c After 8 years

26 a 21 800

b $536

c Year 1: $1 000 000, Year 15: $11 684 800

EXERCISE 6.3

1 a Geometric; $a = 3$, $r = 2$

b Geometric; $a = \frac{1}{2}$, $r = 2\frac{1}{2}$

c Not geometric

d Geometric; $a = \frac{1}{2}, r = \frac{2}{5}$

2 a $c = 36$

b $g = -6, h = 12$

c $p = 2.4, q = 12, s = 60$

3 a $t_n = -1 \times 5^{n-1}$

b $t_n = 7 \times (-0.5)^{n-1}$

c $t_n = \frac{5}{6} \times \left(\frac{2}{3}\right)^{n-1}$

4 a $-2, -6, -18, -54, -162$

b $4, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \frac{4}{81}$

c $\frac{1}{4}, -\frac{3}{8}, \frac{9}{16}, -\frac{27}{32}, \frac{81}{64}$

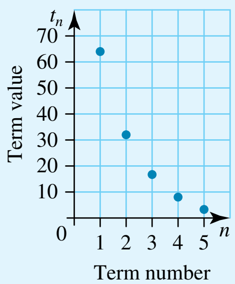
5 a 19 131 876 **b** 5 **c** −420

6 a 1 258 291.2 **b** −2 **c** 640

7 a

Term number	1	2	3	4	5
Term value	64	32	16	8	4

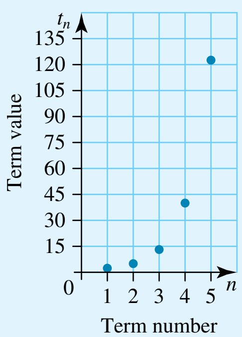
b



8 a

Term number	1	2	3	4	5
Term value	1.5	4.5	13.5	40.5	121.5

b



9 a $t_n = 2507.5 \times 1.003^{n-1}$

b $2507.50, $2515.02, $2522.57, $2530.14, $2537.73, $2545.34

c $2614.89

10 a $4500

b 4.2%

11 a $t_n = 1367.1 \times 0.93^{n-1}$

b $822.59

12 a $1800 **b** 7.5%

13 a Geometric; first term = 3, common ratio = 5

b Not geometric

c Geometric; first term = −8, common ratio = −3

d Geometric; first term = 128, common ratio = $\frac{1}{4}$

e Not geometric

f Geometric; first term = 3, common ratio = $\sqrt{3}$

14 a 7 **b** 4 **c** 4

15 a 1, 3, 9, 27

b 13.24, 66.2, 331, 1655

16 2nd term = 36, 3rd term = 18

17 2nd term = 80, 3rd term = 32, 4th term = 12.8, 5th term = 5.12

18 23

19 a $t_n = 55\,000 \times 1.03^{n-1}$

b $61 902.98

20 a The second value could be either positive or negative.

b 3159 and −3159

21 $10 028.87

22 a $t_n = 641.25 \times 0.95^{n-1}$, where n is the number of 5-km increments of the descent

b 241.98 g

23 a 13 122

b $t_n = 13\,122 \times \left(\frac{1}{3}\right)^{n-1}$

c 19 682

24 a $t_n = 4\,463\,100 \times 1.026^{n-1}$

b $t_n = 4\,729\,050 \times 1.017^{n-1}$

c 8 years

EXERCISE 6.4

1 24, 20, 18, 17, 16.5

2 2, 2, 2, 2, 2

3 $t_{n+1} = t_n - 5, t_1 = 2$

4 −2.2, 1.3, 4.8, 8.3, 11.8

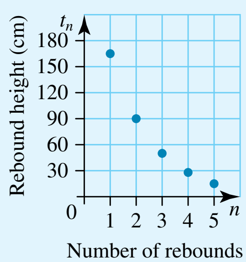
5 $t_{n+1} = -3t_n, t_1 = 2.5$

6 −4, 14, −49, 171.5, −600.25

7 a $t_{n+1} = 0.55t_n, t_1 = 165$

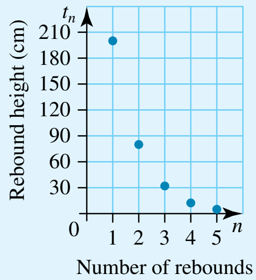
b 4th rebound: 27.45 cm, 5th rebound: 15.10 cm

c



8 a $t_{n+1} = \frac{2}{5}t_n, t_1 = 200$

b 1st rebound: 200 cm, 2nd rebound: 80 cm, 3rd rebound: 32 cm, 4th rebound: 12.8 cm, 5th rebound: 5.12 cm c



9 3, −5, −2, −7, −9, −16, −25, −41

10 2, 1, 3, 4, 7, 11, 18, 29, 47, 76

11 $t_{n+1} = t_n - 4.5, t_1 = 2$

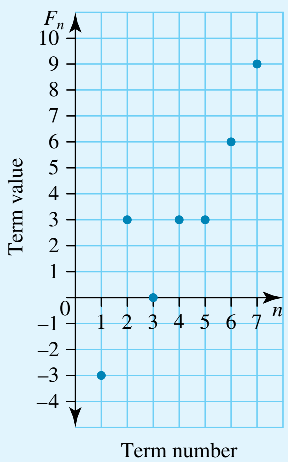
12 $t_{n+1} = \frac{1}{4}t_n, t_1 = -256$

13 $F_{n+2} = F_n + F_{n+1}, F_1 = -10, F_2 = 8$

14 a $t_{n+1} = t_n + 63, t_1 = 18\,063$

b $18 441

15



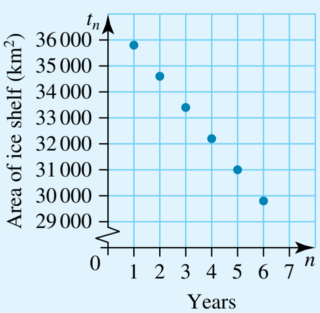
16 a $t_{n+1} = 1.05t_n, t_1 = 6300$

b $7657.69

17 a $t_{n+1} = t_n - 1200, t_1 = 35\,800$

b 35 800 km^2, 34 600 km^2, 33 400 km^2, 32 200 km^2, 31 000 km^2, 29 800 km^2

c



18 3, 9, 15, 33, 63, 129, 255, 513, 1023, 2049

19 a 17.7 metres

b $t_{n+1} = 0.7t_n, t_1 = 12.39$

20 a 5000 rabbits

b $F_{n+2} = F_n + F_{n+1}, F_1 = 5000, F_2 = 15\,000$

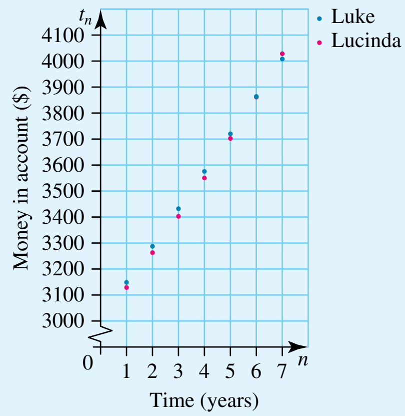
c No, there will be a natural limit to the population of the rabbits depending on resources such as food.

21 a $t_{n+1} = t_n + 144, t_1 = 3144$

b $t_{n+1} = 1.043t_n, t_1 = 3129$

c Luke: $3144, $3288, $3432, $3576, $3720, $3864, $4008
Lucinda: $3129, $3263.55, $3403.88, $3550.25, $3702.91, $3862.14, $4028.21

d



22 Yes, this statement is true provided both of the starting numbers are not 0.