SESSION ONE

Two subjects, Pure Mathematics and Applied Mathematics, are offered in the first semester of a university science course.
The matrix N lists the number of students enrolled in each subject.

$$N = \begin{array}{c} \begin{bmatrix} 540 \\ 360 \end{bmatrix} \end{array} \begin{array}{l} \textit{Pure} \\ \textit{Applied} \end{array}$$

The matrix G lists the proportion of these students expected to be awarded grades A, B, C, D or E in each subject.

$$G = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0.05 & 0.125 & 0.175 & 0.45 & 0.20 \end{bmatrix} \end{array}$$

a. Write down the orders of matrices N and P.

N is a 2×1 matrix. *P is a 1×5 matrix.*

b. The final results matrix (R) shows the final number of students achieving each grade for each Mathematics subject, and is calculated as follows : $R = NG$.

i. Evaluate the matrix R, showing results correct to the nearest integer.

$$R = NG = \begin{bmatrix} 27 & 81 & 189 & 135 & 108 \\ 18 & 54 & 126 & 90 & 72 \end{bmatrix} \begin{array}{l} \textit{Pure} \\ \textit{Applied} \end{array}$$

ii. Explain what the matrix **element** R_{13} represents.

R_{13} represents the number of students who obtained a C in Pure Mathematics.

iii. If an E grade is considered a fail, how many students, in total, will have failed these subjects at the end of the semester.

Total number of failures $= 108 + 72 = 180$

c. Students enrolled in Pure Mathematics have to pay a course fee of $110, while students enrolled in Applied Mathematics pay a course fee of $150. These course fees cover the course of printed notes and access to a variety of resources.

i. Write down a clearly labelled row matrix, called *F*, that lists these fees.

$$F = \begin{array}{c} \begin{array}{cc} P & A \end{array} \\ \begin{bmatrix} 110 & 150 \end{bmatrix} \end{array}$$

ii. Show a matrix calculation that will give the total course fees, *C*, paid in dollars by the students enrolled in Pure and Applied Mathematics.
Find this amount.

$$C = F \times N = \begin{bmatrix} 110 & 150 \end{bmatrix} \times \begin{bmatrix} 540 \\ 360 \end{bmatrix} = \begin{bmatrix} 113400 \end{bmatrix}$$

Course fees total $113 400.

The following transition matrix, *T*, is used to help predict class attendance of Pure Mathematics students at the university on a lecture-by-lecture basis.

$$\begin{array}{ccccc} & \multicolumn{2}{c}{\text{this lecture}} & & \\ & \textit{attend} & \textit{not attend} & & \\ T = & \left[\begin{array}{c} 0.85 \\ 0.15 \end{array}\right. & \left.\begin{array}{c} 0.25 \\ 0.75 \end{array}\right] & \begin{array}{c} \textit{attend} \\ \textit{not attend} \end{array} & \text{next lecture} \end{array}$$

S_0 is the attendance matrix for the Pure Mathematics information session.

$$S_0 = \begin{bmatrix} 510 \\ 30 \end{bmatrix} \begin{array}{l} \textit{attend} \\ \textit{not attend} \end{array}$$

S_0 indicates that 510 Pure Mathematics students attended the information session and 30 Pure Mathematics students did not attend the information session.

d. Use *T* and S_0 to

i. determine S_1 the attendance matrix for the first lecture, writing your answers correct to the nearest integer.

$$S_1 = \begin{bmatrix} 441 \\ 99 \end{bmatrix}$$

ii. predict the number of Pure Mathematics students attending the third lecture, writing your answer correct to the nearest integer.

$$S_3 = \begin{bmatrix} 375 \\ 165 \end{bmatrix}$$

e. Write down a matrix equation for S_n in terms of T, n and S_0, that would enable us to predict the number of students attending any of the 52 lectures scheduled for Pure Mathematics in Semester One.

$$S_n = T^n . S_0$$

To cut down on cleaning costs, the Pure Mathematics lecture will be transferred to a smaller lecture theatre when the number of students predicted to attend falls below 350.

f. For which lecture can this first be done?

$$S_6 = \begin{bmatrix} 346 \\ 194 \end{bmatrix}$$

g. Predict, correct to the nearest integer, the number of students who will be attending the Pure Mathematics lectures :

i. one-quarter of the way through the course

$$S_{13} = T^{12}. S_1 = \begin{bmatrix} 338 \\ 202 \end{bmatrix}$$

ii. one-half of the way through the course

$$S_{26} = T^{25}. S_1 = \begin{bmatrix} 338 \\ 202 \end{bmatrix}$$

iii. three-quarters of the way through the course

$$S_{39} = T^{38}. S_1 = \begin{bmatrix} 338 \\ 202 \end{bmatrix}$$

iv. for the last lecture

$$S_{52} = T^{51}. S_1 = \begin{bmatrix} 338 \\ 202 \end{bmatrix}$$

h. Comment on the results you have found in part g.
Were all the calculations necessary ?

NOT all calculations were necessary, as halfway through the course, student numbers had hit steady state!

SESSION TWO

PROBLEM ONE

The bookshop manager at the university has developed a matrix formula for determining the number of Pure and Applied Mathematics textbooks he should order each year.
For 2016, the starting point for the formula is the column matrix S_0. This lists the number of Pure and Applied Mathematics textbooks sold in 2015.

$$S_0 = \begin{bmatrix} 505 \\ 316 \end{bmatrix} \begin{array}{l} \textit{Pure} \\ \textit{Applied} \end{array}$$

O_1 is a column matrix listing the number of Mathematics and Physics textbooks to be ordered for 2016.

O_1 is given by the matrix formula

$O_1 = A\ S_0 + B$ where $A = \begin{bmatrix} 0.85 & 0 \\ 0 & 0.78 \end{bmatrix}$ and $B = \begin{bmatrix} 80 \\ 75 \end{bmatrix}$

a. Determine O_1 , correct to the nearest integer.

$$O_1 = \begin{bmatrix} 509 \\ 321 \end{bmatrix} \begin{array}{l} \textit{Pure} \\ \textit{Applied} \end{array}$$

b. Given that $S_1 = \begin{bmatrix} 499 \\ 303 \end{bmatrix} \begin{array}{l} \textit{Pure} \\ \textit{Applied} \end{array}$, representing the number of textbooks sold in 2016, determine O_2 (the numbers of books to be ordered for 2017) using the same matrix equation.

$$O_2 = \begin{bmatrix} 504 \\ 311 \end{bmatrix} \begin{array}{l} \textit{Pure} \\ \textit{Applied} \end{array}$$

The matrix formula above only allows the manager to predict the number of books he should order one year ahead. A new matrix formula enables him to determine the number of books to be ordered two or more years ahead.

The new matrix formula is $O_{n+1} = C\,O_n + D$ where O_n is a column matrix listing the number of Pure and Applied Mathematics textbooks to be ordered for year *n*.

For this matrix equation, $C = \begin{bmatrix} 0.85 & 0 \\ 0 & 0.85 \end{bmatrix}$ and $D = \begin{bmatrix} 72 \\ 50 \end{bmatrix}$

The number of books ordered in 2015 was given by

$$O_1 = \begin{array}{c} \begin{bmatrix} 500 \\ 320 \end{bmatrix} \end{array} \begin{array}{l} \textit{Pure} \\ \textit{Applied} \end{array}$$

c. Use the new matrix formula to predict, correct to the nearest integer, the number of each Mathematics textbook the bookshop manager should order in the years 2016 – 2019 (inclusive).

$$\begin{array}{ccccc} O_2 & O_3 & O_4 & O_5 & \\ \begin{bmatrix} 497 \\ 322 \end{bmatrix} & \begin{bmatrix} 494 \\ 324 \end{bmatrix} & \begin{bmatrix} 492 \\ 325 \end{bmatrix} & \begin{bmatrix} 490 \\ 326 \end{bmatrix} & \begin{array}{l} \textit{Pure} \\ \textit{Applied} \end{array} \end{array}$$

d. What do these predictions tell us about the expected popularity of these two subjects in the forecast period ?

Pure Mathematics slightly LESS popular, Applied Mathematics slightly MORE popular.

PROBLEM TWO

By the end of each academic year, students at the university will have either passed, failed or deferred the year.
Experience has shown that in the Science Faculty :

- 88 % of students who pass this year will also pass next year
- 10 % of students who pass this year will fail next year
- *X* % of students who pass this year will defer next year
- 52 % of students who fail this year will pass next year
- *Y* % of students who fail this year will fail next year
- 4 % of students who fail this year will defer next year
- *Z* % of students who defer this year will pass next year
- 10 % of students who defer this year will fail next year
- 25 % of students who defer this year will defer next year.

a. Construct a complete transition matrix for this situation, entering the correct values for *X*, *Y* and *Z*

$$\begin{array}{ccccc} & \textit{This year} & & & \\ \textit{pass} & \textit{fail} & \textit{defer} & & \\ 0.88 & 0.52 & 0.65 & \textit{pass} & \textit{Next year} \\ 0.10 & 0.44 & 0.10 & \textit{fail} & \\ 0.02 & 0.04 & 0.25 & \textit{defer} & \end{array}$$

Twelve hundred and thirty students began a Science degree in 2014.
By the end of the 2014 academic year, 880 students had passed, 230 had failed, while 120 had deferred the year.
No students have dropped out of the Science degree permanently.

b. Use this information to predict the number of Science students who :

i. by the end of the 2015 academic year will have **deferred** the year.

$$T.S = \begin{bmatrix} 972 \\ 201.2 \\ 56.8 \end{bmatrix} = 57 \text{ (to nearest integer)}$$

ii. by the end of the 2016 academic year will have **deferred** the year.

$$T^2.S = \begin{bmatrix} 996.9 \\ 191.4 \\ 41.7 \end{bmatrix} = 42 \text{ (to nearest integer)}$$

iii. by the end of the 2016 academic year will have **passed** their third successive year (and be eligible for graduation).

$$880 \times 0.88 \times 0.88 = 681$$

The situation is quite different in the Arts Faculty, where experience has shown :

- 75 % of students who pass this year will also pass next year
- 15 % of students who pass this year will fail next year
- 10 % of students who pass this year will defer next year
- 55 % of students who fail this year will pass next year
- 35 % of students who fail this year will fail next year
- 10 % of students who fail this year will defer next year
- 65 % of students who defer this year will pass next year
- 15 % of students who defer this year will fail next year
- 20 % of students who defer this year will defer next year.

Three thousand and ten students began an Arts degree in 2014.
By the end of the 2014 academic year, 1980 students had passed, 730 had failed, while 300 had deferred the year.
No students have dropped out of the Art degree permanently.

c. Use this information to predict the number of Art students who :

i. by the end of the 2016 academic year will have **failed** the year.

$$T^2.S = \begin{bmatrix} 2104.9 \\ 571 \\ 334.1 \end{bmatrix} = 571 \text{ (to nearest integer)}$$

ii. by the end of the 2016 academic year will have **failed** their third successive year (and hence may not be allowed back for 2017).

$730 \times 0.35 \times 0.35 = 89$

SESSION THREE

PROBLEM ONE

Members of the Arts Faculty, love playing around with anagrams – letter combinations that can generate a number of different words. For example, the letters *A*, *C*, *D, E* and *R* can form the words *CADRE*, *CARED*, *CEDAR* or *RACED.*

More mathematically inclined members of the Arts Faculty know that permutation matrices can be used to rearrange the letters in a word.

(a) If matrix $W = \begin{bmatrix} L \\ E \\ A \\ S \\ T \end{bmatrix}$ and matrix $P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, then what word is formed by the matrix product $P \times W$?

$$\begin{bmatrix} T \\ A \\ L \\ E \\ S \end{bmatrix}$$

(b) In the matrix provided below, fill in the element values for matrix *Q* so that the matrix product *Q* × *W* gives the word SLATE.

$$\begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \times \begin{bmatrix} L \\ E \\ A \\ S \\ T \end{bmatrix} = \begin{bmatrix} S \\ L \\ A \\ T \\ E \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} L \\ E \\ A \\ S \\ T \end{bmatrix} = \begin{bmatrix} S \\ L \\ A \\ T \\ E \end{bmatrix}$$

(c) Explain why the matrix product $Q^4 \times W$ gives the matrix $\begin{bmatrix} L \\ E \\ A \\ S \\ T \end{bmatrix}$.

$$Q^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ which is the 5 × 5 identity matrix.

PROBLEM TWO

The five departments in the Science Faculty played a round-robin table tennis tournament, in which every department played against every other department once.
In each game there was a winner and a loser.

A table of one-step and two-step dominances was prepared to summarise the results.

Team	one-step dominances	two-step dominances
Botany (*B*)	1	2
Chemistry (*C*)	3	5
Geology (*G*)	2	4
Physics (*P*)	3	4
Zoology (*Z*)	1	1

(a) Use the one-step and two-step dominance values to construct a one-step dominance matrix, clearly explaining the allocation of values for each team. Label the rows as Winners and the columns as Losers.

The key to answering this question is to realise that if team A beats team B, then team B's one-step dominances becomes team A's two-step dominances. Work through each department's results in the table systematically, starting with Zoology.

Zoology won one game and as a result picked up one two-step dominance.
The only way to do this was that Zoology defeated Botany (which had won one match).

Botany won one game and as a result picked up two two-step dominances.
The only way to do this was that Botany defeated Geology (which had won two matches).

Chemistry won three games and as a result picked up five two-step dominances.
Two of these wins would have been against Botany and Zoology (one two-step and one two-step dominances respectively).
The other three two-step dominances would have been picked up from the defeat of Physics.
So Chemistry defeated Botany, Zoology and Physics.

Physics won three games and as a result picked up four two-step dominances.
Since Physics has lost to Chemistry, it must have beaten Botany, Geology and Zoology (one two-step, two two-step and one two-step dominances respectively.
So Physics defeated Botany, Geology and Zoology.

Geology won two games and as a result picked up four two-step dominances.
Since it has already lost to Botany and Physics, these two wins would have been against Chemistry and Zoology (three two-step and one two-step dominances respectively).
So Geology defeated Chemistry and Zoology.

So the one-step dominance matrix is :

$$D1 = \begin{array}{c} \begin{array}{ccccc} B & C & G & P & Z \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ B \\ C \\ G \\ P \\ Z \end{array}$$

(b) Use the one-step and two-step dominance values to construct a two-step dominance matrix, clearly explaining the allocation of values for each team. Label the rows as Winners and the columns as Losers.

. *Work through each department's results in the table systematically, starting with Zoology.*

Zoology defeated Botany, gaining one two-step dominance from Botany, when it defeated Geology. Zoology gets a "1" in the Geology column

Botany defeated Geology, gaining two two-step dominances from Geology, when it defeated Chemistry and Zoology. Botany gets a "1" in each of the Chemistry and the Zoology columns.

Chemistry defeated Botany, Zoology and Physics, gaining one two-step from Botany, when it defeated Geology, one two-step from Zoology, when it defeated Botany, and three two-step dominances from Physics, when it defeated Botany, Geology and Zoology. Chemistry gains a "2" in the Botany column, a "2" in the Geology column and a "1" in the Zoology column.

Physics defeated Botany, Geology and Zoology, gaining one two-step from Botany, when it defeated Geology, two two-steps from Geology, when it defeated Chemistry and Zoology, and one two-step dominance from Zoology, when it defeated Botany. Physics gains a "1" in each of the Botany, Chemistry, Geology and Zoology columns.

Geology defeated Chemistry and Zoology, gaining three one-step from Chemistry, when it defeated Botany, Physics and Zoology, and one two-step dominance from Zoology when it defeated Biology. Geology gains a "2" in the Botany column, and a "1" in each of the Physics and Zoology columns.

So the two-step dominance matrix is:

$$D2 = \begin{array}{c} \begin{array}{ccccc} B & C & G & P & Z \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ B \\ C \\ G \\ P \\ Z \end{array}$$

(c) Construct the final dominance matrix for this competition by adding together the one-step and two-step dominance matrices.

$$D1+D2=\begin{array}{c} \begin{matrix} B & C & G & P & Z \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 3 & 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \\ B \\ C \\ G \\ P \\ Z \end{matrix}$$

(d) What is the finishing order in this competition ?
Chemistry, with a total of 8, won from Physics (7), Geology (6), Biology (3) and Zoology (2).

(e) Write in the results of every match in the table below.

Round	Participants	Result
1	Biology vs Geology Physics vs Zoology	...........*Biology*........... ....*Physics*...........
2	Chemistry vs Biology Zoology vs Geology	...........*Chemistry*......... ....*Geology*...........
3	Geology vs Chemistry Physics vs Biology	...........*Geology*........... ....*Physics*............
4	Zoology vs Chemistry Geology vs Physics	...........*Chemistry*......... ....*Physics*...........
5	Chemistry vs Physics Biology vs Zoology	...........*Chemistry*......... ....*Zoology*...........

(f) Draw a directed graph which displays the results of this competition, with the arrow pointing towards the loser of each game.