

Year 11

General Mathematics 2017

Strand: Discrete Mathematics

In this area of study students cover matrices, graphs and networks, and number patterns and recursion, and their use to model practical situations and solve a range of related problems.

Chapter 4 – Matrices

This topic includes:

• Use of matrices to store and display information that can be presented in a rectangular array of rows and columns such as databases and links in social and road networks.

• Types of matrices (row, column, square, zero and identity) and the order of a matrix.

• Matrix addition, subtraction, multiplication by a scalar, and matrix multiplication including determining the power of a square matrix using technology as applicable.

• Use of matrices, including matrix products and powers of matrices, to model and solve problems, for example costing or pricing problems, and squaring a matrix to determine the number of ways pairs of people in a network can communicate with each other via a third person.

• Inverse matrices and their applications including solving a system of simultaneous linear equations.

**Key knowledge**

• The concept of a matrix and its use to store, display and manipulate information.

• Types of matrices (row, column, square, zero, identity) and the order of a matrix.

• Matrix arithmetic: the definition of addition, subtraction, multiplication by a scalar, multiplication, the power of a square matrix, and the conditions for their use.

• Determinant and inverse of a matrix.

**Key skills**

• Use matrices to store and display information that can be presented in rows and columns.

• Identify row, column, square, zero, and identity matrices and determine their order.

• Add and subtract matrices, multiply a matrix by a scalar or another matrix, raise a matrix to a power and determine its inverse, using technology as applicable.

• Use matrix sums, difference, products, powers and inverses to model and solve practical problems.

|  |  |
| --- | --- |
| **Chapter Sections** | **Questions to be completed** |
| **4.2** Types of Matrices | 1,2,3,4,5,6,7,8,9,12,15,16,18. |
| **4.3** Operations with Matrices | 1,2,3,4,5,6,8,10,13,14. |
| **4.4** Matrix Multiplication | 1,2,3,4,5,6,7,8,9,0,12,14,15,16,18,20,22. |
| **4.5** Inverse Matrices and problem solving with Matrices | 1,3,4,5,6,7,8,9,10,12,15,16,17,18,19,23,24,25,26. |

# 4.2 Type of Matrices

A matrix (singular) or matrices(plural) is a two dimensional array of numbers arranged in rows and columns that is used to represent many different types of information.

## Describing matrices

A matrix is usually displayed in square brackets with no borders between the rows and columns.

Example 1: The table below shows the number of adults and children who attended three different events over the school holidays. Construct a matrix to represent this information.





## Networks

Matrices can also be used to display information about various types of networks, including road systems and social networks.



Example 2: The distances, in kilometres, along three major roads between the Tasmanian towns Launceston (L), Hobart (H) and Devonport (D) are displayed in the matrix below.



1. What is the distance, in kilometres, between Devonport and Hobart?

1. Victor drove 75 km directly between two of the Tasmanian towns. Which two towns did he drive between?

1. The Goldstein family would like to drive from Hobart to Launceston, and then to Devonport. Determine the total distance in kilometres that they will travel.

## [http://t2.gstatic.com/images?q=tbn:ANd9GcRhsX0jyBK5uasHr2kNMmna6H7YTHHO84V7TlMCFeFb1G-Lgdlq](http://www.google.com.au/url?sa=i&source=images&cd=&cad=rja&docid=Ffh7CV4CK0AasM&tbnid=d5WSEpAFq27f7M:&ved=0CAgQjRwwAA&url=http://en.wikipedia.org/wiki/Column&ei=fjY0UqrDEuGriAel1IG4DQ&psig=AFQjCNEoSQjdo3SScoyTtAtajMSSwGfZhQ&ust=1379239934380954)Defining Matrices

Matrices are represented by rows and columns.

1st Column

2nd Column

1st Row

2nd Row

3rd Row

This matrix has 3 rows and 2 columns and is called a (3 × 2) matrix

This is a (3 × 3) matrix as it has 3 rows and 3 columns

Generally a matrix with *m* rows and *n* columns is called an (*m* × *n*) matrix. This is also referred as the order/size/dimension of the matrix.

A matrix that has the same number of rows and columns is called a square matrix.

|  |  |  |
| --- | --- | --- |
|  |  |  |
| (2 × 2) | (3 × 3) | (4 × 4) |

A row matrix has only one row.

A column matrix has only one column.

Example 3: At High Vale College, 150 students are studying General Mathematics and 85 students are studying Mathematical Methods. Construct a column matrix to represent the number of students studying General Mathematics and Mathematical Methods, and state the order of the matrix.

General Maths students = \_\_\_\_\_

Maths Methods students = \_\_\_\_\_

The order of the matrix is therefore \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.

## Elements of matrices

The entries in a matrix are called elements. The position of an element is described by the corresponding row and column. For example, *a*21 means the entry in the 2nd row and 1st column of matrix *A*, as shown below.

1st Column



2nd Row

Note matrices are often named by alphabet using the upper case, whereas its elements are denoted by the same letter but using the lower case.

Example: For each of the following, give the order of the matrix, if it exists. Where possible, write the 2, 1 element and 1, 3 element of each.

|  |  |  |  |
| --- | --- | --- | --- |
| **Matrix** | **Order** | **2,1 element** | **1,3 element** |
| A |  |  |  |
| B |  |  |  |
| C |  |  |  |
| D |  | | |

## Identity matrices

An identity matrix is denoted by the symbol *I*. It is a square matrix where all of the elements on the diagonal line from the top left to bottom right are 1s and all of the other elements are 0s.

Example:

and are both identity matrices.

## The Zero matrix

A zero matrix, *0*, is a square matrix that consists entirely of ‘0’ elements.

These two matrices are examples of zero matrices: ,

# 4.3 Operations with matrices

## Matrix addition and subtraction

Matrices can be added and subtracted if they have the same order/size/dimension. That is if they have the same number of rows and columns.

### Adding matrices

To add matrices, add the corresponding elements of each matrix together. That is the numbers in the same position.

### Subtracting matrices

To subtract matrices, subtract the corresponding elements of each matrix together. That is the numbers in the same position.

Example:

*A* = *B* = *C* =

Find if possible

1. *A* + *B* =
2. *A* − *B* =
3. *B* − *C* =

Alternatively, we can use CAS calculator to perform matrix addition and subtraction.

|  |  |
| --- | --- |
| Define matrices *A* and *B*  On a calculator page, press:   * MENU b * 1: Actions 1 * 1: Define 1   The template for the (2 × 2) can be found by pressing / t  Complete the entry lines as:   * Define *a* = * Define *b* = * Define c =   Press ENTER after each entry |  |
| Complete the entry lines as:   * *a* + *b* * *a* – *b* * *a* – *c*   Press ENTER after each entry  Notices the subtraction between the two matrices *A* – *C* is not possible as these two matrices do not have the same dimension, therefore the calculator is returning  “**Error: Dimension mismatch**” |  |

# 4.4 Matrix multiplication

## Multiplying by a scalar

If matrix, then A + A can be found by adding A to itself using matrix addition or more easier multiply matrix A by 2. This is because A + A = 2A



This is known as scalar multiplication as the number 2, a scalar quantity, is multiplied with a matrix. Any matrix can be multiplied by any scalar quantity and the order of the matrix will remain the same.

Multiplying by a scalar means every entry in the matrix is multiplied by the same number.

Example:

If find:

1. 2A =

Alternatively, we can use CAS calculator to perform scalar multiplication

|  |  |
| --- | --- |
| Define matrices *A* and *B*  On a calculator page, press:   * MENU b * 1: Actions 1 * 1: Define 1   The template for the (2 × 2) can be found by pressing / t  Complete the entry lines as:   * Define *a* = * Define *b* =   Press ENTER after each entry  Complete the entry lines as:   * 2*a*   Press ENTER after each entry |  |

## The product matrix and its order

Not all matrices can be multiplied together. However, unlike with addition and subtraction, matrices do **not** need to have the same order to be multiplied together. For matrices to be able to be multiplied together (have a product), the *number of columns* in the first matrix must equal the *number of rows* in the second matrix.

Example:

|  |  |  |  |
| --- | --- | --- | --- |
| Multiplication of matrix test | A | B | AB |
| Matrix order |  |  |  |

Since the two middle number are the \_\_\_\_\_\_\_\_\_\_, it is possible to \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ the two matrices \_\_\_\_\_ and \_\_\_\_\_ together. That is the \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ of matrix AB exists.

When the product of two matrix exits, it is possible to work out the order of the resultant or the answer matrix in advance. From the above example, the product of A and B will have an order of \_\_\_\_\_\_\_\_\_\_. These numbers are obtained from the two outside numbers from the table above.

Example 9: If A= and , determine the product matrix AB.

|  |  |  |  |
| --- | --- | --- | --- |
| Multiplication of matrix test | *A* | *B* | *AB* |
| Matrix order |  |  |  |

Example 10: Determine the product matrix *MN* if and .

|  |  |  |  |
| --- | --- | --- | --- |
| Multiplication of matrix test | *M* | *N* | *MN* |
| Matrix order |  |  |  |

## Multiplying by the Identity matrix (Unit matrix)

Given two matrices A and I below

1. Calculate *AI*
2. Calculate *IA*

When multiplying any matrix by an identity matrix the resultant matrix will be the original matrix.

*AI* = *IA* = *A*

## Powers of square matrices

When a square matrix is multiplied by itself or raised to the power of a whole number, the order of the resultant matrix is equal to the order of the original square matrix.

Example 11: If , calculate the value of *A*3.

*A*3 =

Notice that the order of the resultant matrix will always have the same order as the original matrix *A*, that is \_\_\_\_\_\_\_\_\_\_ in this case.

# 4.5 Inverse matrices and problem solving with matrices

## Inverse matrices

In the real number system, a number multiplied by its reciprocal results in 1. For example, .In this case is the reciprocal or multiplicative inverse of 3.

In matrices, if the product matrix is the identity matrix, then one of the matrices is the multiplicative inverse of the other.

Example: If is multiplied by the matrix

Then the matrix is the multiplicative inverse of A. The inverse of A is denoted by A−1.

Now let examine the reverse multiplication of A and its inverse.

From above we can see that *A.A*−1 *= I*

*and A*−1*.A = I*

Therefore *A.A*−1 *= I = A*−1*.A*

Example 12: By finding the product matrix *AB*, determine whether the following matrices are multiplicative inverses of each other.

Since the product of *AB* gives an \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ matrix, therefore matrix *A* and *B* are multiplicative \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ of each other.

## Finding the inverse matrices

Inverse matrices only exist for square matrices and can be easily found for matrices of order 2×2. Inverses can also be found for larger square matrices; however, the processes to find these are more complicated, so technology (CAS calculator) is often used to find larger inverses.

Example: Determine the inverse for the matrix

Step 1: Determine the determinant of matrix *A*

Det A = = (1 × 4) − (2 × −3) = 4 + 6 = 10

Step 2: Swap the elements in the main the diagonal

Step 3: Multiply the elements on the other diagonal by −1 or simply swap the signs of these numbers

Step 4: Write the inverse matrix of *A*

Determinant

Example 13: If , determinant *A*−1

Calculate the determinant of *A* =

Calculate the inverse of *A*−1 =

Alternatively, we can use CAS calculator to calculate the determinant and the inverse of a matrix

|  |  |
| --- | --- |
| Define matrices *A*  On a calculator page, enter as follows to determine the determinant of matrix A   * det(*a*) or det |  |
| Define matrices *A*  On a calculator page, enter as follows to determine the determinant the inverse of matrix A   * ***a***−1 or |  |

Singular Matrix

Example: Determine the inverse for the matrix B =

Calculate the determinant of *B* =

Calculate the inverse of *B*−1 =

Since the determinant of matrix B is equal to \_\_\_\_\_, therefore the inverse of B does not \_\_\_\_\_.

Matrix B is said to be \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ matrix.

## Using inverse matrices to solve problems

Unlike in the real number system, we can’t divide one matrix by another matrix. However, we can use inverse matrices to help us solve matrix equations in the same way that division is used to help solve many linear equations.

Given the matrix equation *AX* = *B*, where matrix *X* is the unknown. We can use the multiplication of the inverse matrix to find the value of the unknown matrix *X* as follows:

If the equation was *XA = B*, where the matrix *X* is the unknown that isneeded to be found. Again we can apply the principal of inverse multiplication to determine for the value of matrix *X*:

Example 14: If , find the value of *x* and *y*.

The matrix equation above is presented in the form \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Where *A* = *X* = *B* =

Calculate for the determinant of matrix *A* =

Calculate the inverse of matrix *A* =

Solve for the matrix *X* =

## Using inverse matrices to solve a system of simultaneous equations

If you have a pair of simultaneous equations, they can be set up as a matrix equation and solved using inverse matrices.

Example 15: Solve the following pair of simultaneous equations by using inverse matrices.

2*x* + 3*y* = 6

4*x* – 6*y* = −4

Set up the simultaneous equations as a matrix equation then solve for the value of the unknowns

## Using matrix equations to solve worded problems

To use matrices to solve worded problems, you must set up a matrix equation from the information provided. The matrix equation can then be solved using the inverse multiplication.

Example 16: On an excursion, a group of students and teachers travelled to the city by train and returned by bus. On the train, the cost of a student ticket was $3 and the cost of a teacher ticket was $4.50, with the total cost for the train tickets being $148.50. On the bus, the cost of a student ticket was $2.75 and the cost of a teacher ticket was $3.95, with the total cost for the bus tickets being $135.60. By solving a matrix equation, determine how many students and teachers attended the excursion.

Identify the two unknowns in the problem and assign a pronumeral to represent each unknown.

## Adjacency matrices

Matrices can be used to determine the number of different connections between objects, such as towns or people. They can also be used to represent tournament outcomes and determine overall winners. To determine the number of connections between objects, a matrix known as an adjacency matrix is set up to represent these connections.

Example 17: The diagram at right shows the number of roads connecting between four towns, A, B, C and D. Construct an adjacency matrix to represent this information.



|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | TO | | | |  |  |  |  |  |  |  |  |  |
|  |  | A | B | C | D |  |  |  |  |  |  |  |  |  |
| F | A |  |  |  |  |  |  |  |  |  |  |  |
| R | B |  |  |  |  |  |  |  |  |  |  |  |
| O | C |  |  |  |  |  |  |  |  |  |  |  |
| M | D |  |  |  |  |  |  |  |  |  |  |  |

## Determine the number of connections between objects

An adjacency matrix allows us to determine the number of connections between objects. A one-step adjacency matrix means a direct connection between two objects, while a two-steps adjacency matrix means a connection between two objects via a third object, for example the number of ways a person can travel between towns A and D via another town.

You can determine the number of connections of differing ‘steps’ by raising the adjacency matrix to the power that reflects the number of steps in the connection.

For example, the following diagram shows the number of roads connecting five towns, A, B, C, D and E. There are a number of ways to travel between towns A and D. There is one direct path between the towns; this is a one-step path. However, you can also travel between towns A and D via town C or B. These are considered two-step paths as there are two links (or roads) in these paths. The power on the adjacency matrix would therefore be 2 in this case.

Use matrix to represent a one-step path for the above network (denoted as *A*)

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | TO | | | |  |  |  |  |  |  |  |  |  |
|  |  | A | B | C | D | E |  |  |  |  |  |  |  |  |
| F | A |  |  |  |  |  |  |  |  |  |  |  |  |
| R | B |  |  |  |  |  | *A* = |  |  |  |  |  |  |
| O | C |  |  |  |  |  |  |  |  |  |  |  |  |
| M | D |  |  |  |  |  |  |  |  |  |  |  |  |
|  | E |  |  |  |  |  |  |  |  |  |  |  |  |

Use matrix to represent a two-steps path for the above network (denoted as *A*2)

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | TO | | | |  |  |  |  |  |  |  |  |  |
|  |  | A | B | C | D | E |  |  |  |  |  |  |  |  |
| F | A |  |  |  |  |  |  |  |  |  |  |  |  |
| R | B |  |  |  |  |  |  | *A*2 = |  |  |  |  |  |
| O | C |  |  |  |  |  |  |  |  |  |  |  |  |
| M | D |  |  |  |  |  |  |  |  |  |  |  |  |
|  | E |  |  |  |  |  |  |  |  |  |  |  |  |

Notice a two-steps path can be obtained simply by squaring the one-step path matrix *A*,

Therefore a three-steps path matrix use to represent the above network can be obtained by raising the one-step path matrix *A* to the power of \_\_\_\_\_, that is \_\_\_\_\_.

Example 18: The following adjacency matrix shows the number of pathways between four attractions at the zoo: Lions (L), Seals (S), Monkeys (M) and Elephants (E).



Using CAS or otherwise, determine how many ways a family can travel from the Lions to the Monkeys via one of the other two attractions.

Solution

The number of ways a family can travel from the Lions to the Monkeys via one of the other two attractions indicates that we need to determine a \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ path matrix. A simple way of determining a two-steps path matrix is simply raise the matrix to the power of \_\_\_\_\_\_.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |  |
| *A*2 | = |  |  |  |  | = |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |

Basing on the \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ path (*A*2) matrix, we can say that there are \_\_\_\_\_ ways in which a family can travel from the Lions to the Monkeys via one of the other two attractions.

**Transition Matrices Applications**

Example 19:

In a large country town, there are three major supermarkets. Customers switch from one to another due to advertising, better service, prices and for other reasons. A survey of 1000 customers has revealed the following information for the past month.

Best buys started with 40% of the market; 90% of its customers remained loyal to Best Buys but 5% changed to Great Groceries and 5% to Super Store.

Great Groceries started with a 36% market share: 85% remained loyal, 10% transferred to Best Buys and 5% to Super Store.

Super Store stared with 24% of the customers: it lost 15% to Best Buys and 5% to Great Groceries, but 80% remained.

Summarise the information in matrix form and calculate the new market share.

Example 20:

At a large retail outlet, 60% of people drink coffee and 40% drink tea. The catering company has decided to introduce a new brand of coffee and market research shows that of those who drink tea 45% will change to coffee each week and of those who drink coffee only 10% will change to tea each week. The remainder will continue to drink the same drink as present.

1. Draw a tree diagram to represent this situation for week 1.
2. What proportion of people will drink coffee at the end of week 1?
3. Set up matrices to represent the situation.
4. Solve the matrices to show that you get the same answer as in part (b).
5. What proportion of people will drink coffee at the end of week 3?

Transition Matrices Questions

**Question 1**

At a large retail outlet, 55% of people drink coffee and 45% drink tea. The catering company has introduced a new brand of tea and market research shows that of those who drink tea 15% will change to coffee each week and of those who drink coffee 75% will change to tea each week.

1. Draw a tree diagram to represent this situation for 1 week.
2. What proportion of people will drink coffee at the end of 1 week?
3. Set up matrices to represent this situation.
4. Solve the matrices to show that you get the same answer as in part **(b)**.

**Question 2**

At a large retail outlet, only two types of milkshake are produced. At the moment, 45% of people drink chocolate milkshakes and 55% drink strawberry milkshakes. The catering company has decided to introduce a richer strawberry milkshake in place of the current one, and market research shows that of those who drink chocolate milkshakes 35% will change to strawberry each month and of those who drink strawberry only 5% will change to chocolate each month.

1. What is the initial state matrix?
2. What is the transition matrix?
3. What proportion of people will drink each type of milkshake at the end of 1 month?
4. What proportion of people will drink each type of milkshake at the end of 2 months?
5. What proportion of people will drink each type of milkshake at the end of 3 months?
6. What proportion of people will drink each type of milkshake at the end of 100 months?
7. What proportion of people will drink each type of milkshake at the end of 101 months?
8. What do you notice about the answers for (f) and (g)?

**Question 3**

24% of students in a large school own a Warren mobile phone and the rest own an Oval mobile phone. The company that owns Warren decided to run a series of advertisements to promote Warren and market research shows that 15% of students who own an Oval mobile phone will change to Warren each month and 10% of students who own a Warren mobile phone will change to Oval each month. Assume they are all on monthly plans.

1. What is the initial state matrix?
2. What is the transition matrix?
3. What proportion of the students will use each type of phone at the end of 1 month?
4. What proportion of the students will use each type of phone at the end of 2 months?
5. What proportion of the students will use each type of phone at the end of 3 months?
6. What proportion of the students will use each type of phone at the end of 50 months? Give your answer correct to 2 decimal places.
7. What proportion of people will students will use each type of phone at the end of 51 months? Give your answer correct to 2 decimal places.
8. What do you notice about the answers for (f) and (g)?

**Question 4**

35% of students travel by train to a certain school and the rest travel by bus. Vic Rail decided to offer a huge discount to school students to increase their market share. It is known that 20% of students who travel by bus will switch to travelling by train and only 5% of students who travel by train will switch to travelling by bus.

1. What is the initial state matrix?
2. What is the transition matrix?
3. What proportion of the students will use each type of transport at the end of 1 month?
4. What proportion of the students will use each type of transport at the end of 2 months?
5. What proportion of the students will use each type of transport at the end of 3 months?
6. What proportion of the students will use each type of transport at the end of 3 years? Give your answer correct to 2 decimal places.