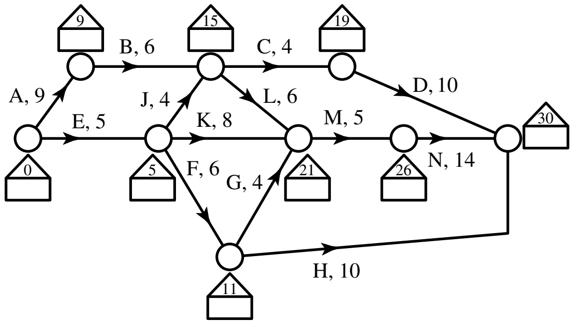
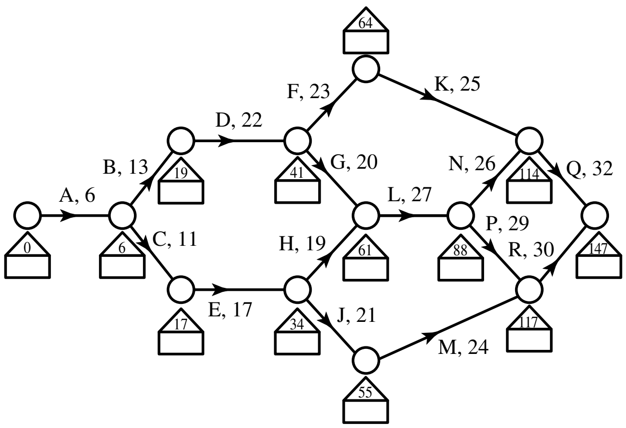
Topic 10 — Directed graphs and networks

Exercise 10.2 — Critical path analysis

1 Earliest completion time is 30 minutes.



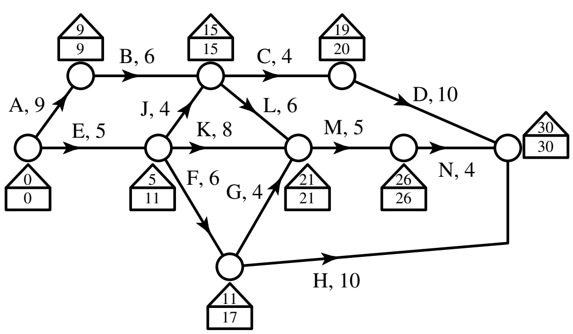
2 The earliest completion time = 147 days.



3 With an earliest completion time of 30:
$A—B—L—M—N$
$A—B = 9+6 = 15$ minutes, compared to
$E—J = 5+4 = 9$ minutes
So a difference of $15-9 = 6$
Thus J can increase by 6; $4+6 = 10$
J can increase to 10.

4 With an earliest completion time of 147:
$A—B—D—G—L—P—R$
$G—L—P—R = 20+27+29+30 = 106$ davs
$F—K—Q = 23+25+32 = 80$ days
So a difference of $106-80 = 26$
If path F and K stay the same, Q can increase by 26
$Q: 32+26 = 58$ days

5



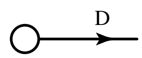
a Critical path = A—B—L—M—N
b Float(D) = 30 − 19 − 10 = 1
Float(C) = 20 − 15 − 4 = 1
Float(H) = 30 − 11 − 10 = 9
Float(F) = 17 − 5 − 6 = 6
Float(G) = 21 − 11 − 4 = 6
Float(E) = 11 − 0 − 5 = 6
Float(K) = 21 − 5 − 8 = 8
Float(J) = 15 − 5 − 4 = 6

6 Float(C) = 25 − 6 − 11 = 8 days
Float(E) = 42 − 17 − 17 = 8 days
Float(F) = 90 − 41 − 23 = 26 days
Float(H) = 61 − 34 − 19 = 8 days
Float(J) = 93 − 34 − 21 = 38 days
Float(K) = 115 − 64 − 25 = 26 days
Float(M) = 117 − 55 − 24 = 38 days
Float(N) = 115 − 88 − 26 = 1 day
Float(Q) = 147 − 114 − 32 = 1 day

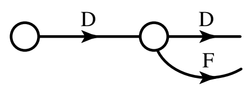
7

Activity	Immediate predecessor
D	—
E	D
F	D
G	E, F

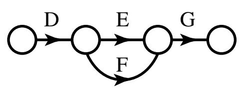
Activity D has no predecessor and is the first edge.



Activity E and F have D as an immediate predecessor.



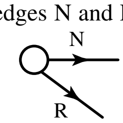
Activity G has E and F as immediate predecessors.



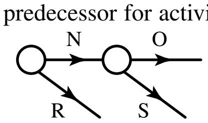
8

Activity	Immediate predecessor
N	—
O	N
P	O, T
Q	P
R	—
S	N
T	S, Y
U	O, T
V	O, T
W	V
X	Y
Y	R
Z	X

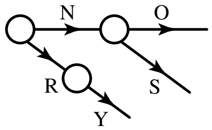
First edges N and R have no immediate predecessors.



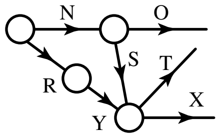
N is a predecessor for activity O and S.



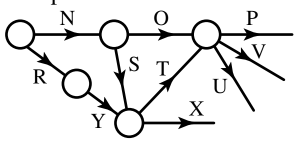
R is a predecessor for activity Y.



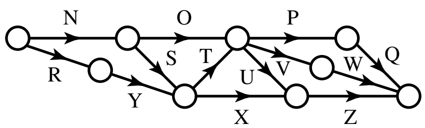
Y is a precedessor for activity X, S and Y are precedessors for activity T.



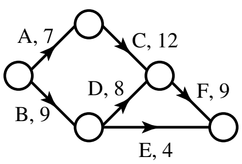
O and T are precedessors for activities P, U and V.



P is a precedessor for activity Q, V is a precedessor for activity W.



9 a



Which of the following statements is true?

A Activity A is an immediate predecessor of F. — False, A is a predecessor for C.

B Activity D is an immediate predecessor of F. — True.

C Activity F must be done before activity D. — False, F after D.

D Activity F must be done before activity E. — False.

E Activity D is an immediate predecessor of E. — No, can occur simultaneously.

The answer is **B**.

b Minimum time to complete all activities follows path

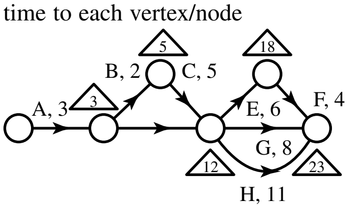
A—C—F

= 7 + 12 + 9

= 28 minutes

The answer is **D**.

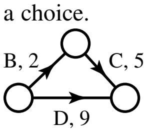
10 a Earliest completion time — fill in triangles with maximum time to each vertex/node



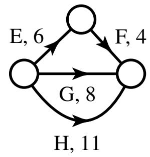
The earliest completion time is 23 minutes.

b Tasks which can be delayed.

Identify sections of the network where there was a choice.



and



$B—C = 2 + 5 = 7$ mins

$D = 9$ mins

$\therefore$ B and C can be delayed.

$E—F = 6 + 4 = 10$ mins

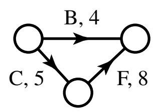
$G = 8$ mins

$H = 11$ mins

$\therefore$ E, F, and G can be delayed.

11 The critical path for the network is A—C—F.

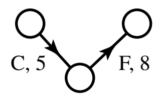
12 a



Select maximum value. 5 + 8 = 13

The answer is **E**.

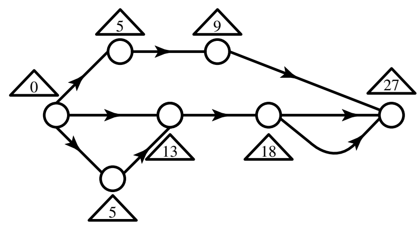
b



13 + 5 = 18

The answer is **D**.

c The earliest completion time for all tasks is:



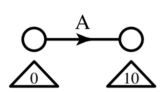
A—D—G

= 3 + 6 + 18

= 27 minutes

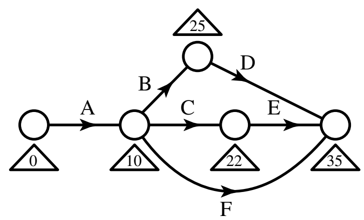
The answer is **A**.

13 a



A—B = 10 + 15 = 25

A—C = 10 + 12 = 22

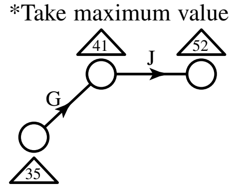


$A—B—D = 25 + 8 = 33$

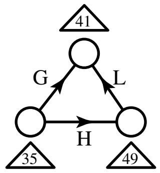
$A—C—E = 22 + 10 = 32$

$A—F = 10 + 25 = 35$

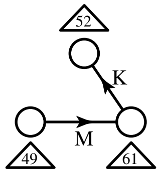
*Take maximum value



$$\begin{aligned} 35\text{—}G &= 35+6 \\ &= 41 \\ 35\text{—}G\text{—}J &= 41+11 \\ &= 52 \end{aligned}$$

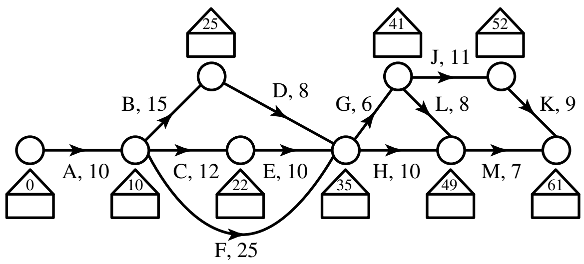


$$\begin{aligned} 35\text{—}G\text{—}L &= 35+6+8 \\ &= 49 \quad \text{* Take maximum value} \\ 35\text{—}H &= 35+10 \\ &= 45 \end{aligned}$$



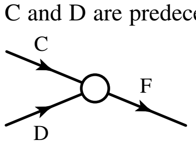
$$\begin{aligned} 52\text{—}K &= 52+9 \quad \text{* Take maximum value} \\ &= 61 \\ 49\text{—}M &= 49+7 \\ &= 56 \end{aligned}$$

FINAL OUTCOME:

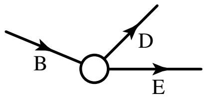


b The earliest completion time for the project
A—F—G—J—K
= 10 + 25 + 6 + 11 + 9
= 61 minutes

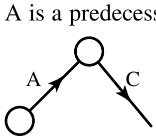
14 Working backwards to find predecessors.
C and D are predecessors of F.



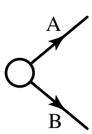
B is a predecessor of E and D.



A is a predecessor of C.

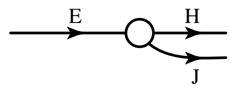


A and B have no predecessors.

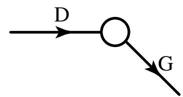


Activity letter	Immediate predecessor	Time
A	—	7
B	—	9
C	A	12
D	B	8
E	B	4
F	C, D	9

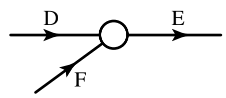
15 Working backwards to find predecessors.
E is a predecessor of H and J.



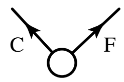
D is a predecessor of G.



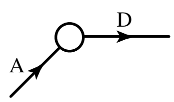
B and F are predecessors of E.



C is a predecessor of F.



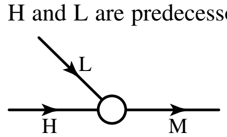
A is a predecessor of D.



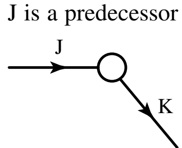
A, B and C have no predecessors.

Activity letter	Immediate predecessor	Time
A	—	3
B	—	4
C	—	5
D	A	6
E	B, F	5
F	C	8
G	D	18
H	E	8
J	E	6

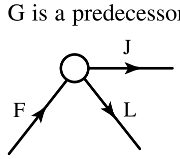
16 Working backwards to find predecessors.
H and L are predecessors of M.



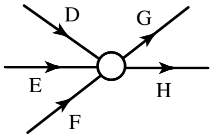
J is a predecessor of K.



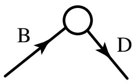
G is a predecessor of J and L.



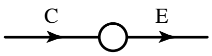
D, E and F are predecessors of G and H.



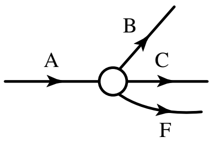
B is a predecessor of D.



C is a predecessor of E.



A is a predecessor of B, C and F.



A has no predecessor.

Activity letter	Immediate predecessor	Time
A	—	10
B	A	15
C	A	12
D	B	8
E	C	10
F	A	25
G	D, E, F	6
H	D, E, F	10
J	G	11
K	J	9
L	G	8
M	H, L	7

17 a Critical path – follows activities that cannot be delayed. The path that takes the largest time is

$$\begin{aligned} A\text{—}D\text{—}G &= 3+6+18 \\ &= 27 \text{ minutes} \end{aligned}$$

b and **c** Float time is the maximum time that an activity can be delayed without delaying a subsequent activity on the critical path.
Activity B can be delayed 10 minutes, Activity C can be delayed 1 minute, Activity E can be delayed 1 minute, Activity F can be delayed 1 minute, Activity H can be delayed 1 minute and Activity J can be delayed 3 minutes.

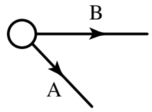
18 a Critical path = longest path
A—F—G—J—K
= 10 + 25 + 6 + 11 + 9
= 61 minutes

b Activities which have float times are not on the critical path. These are therefore B, D, C, E, H, L and M.

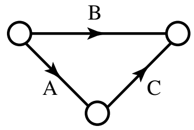
19 a

Activity	Immediate predecessor
A	—
B	—
C	A

Activity A and B have no predecessor, so they become the first edge.



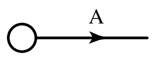
Activity C has A as an immediate predecessor.



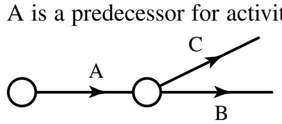
b

Activity	Immediate predecessor
A	—
B	A
C	A
D	C
E	B
F	B
G	F
H	D, E, G
J	D, E, G
I	J, H

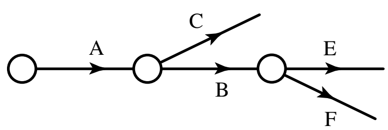
First edge A has no predecessor.



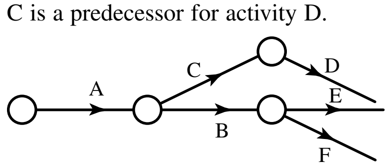
A is a predecessor for activity B and C.



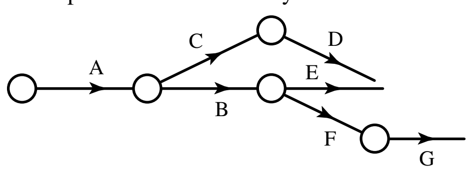
B is a predecessor for E and F.



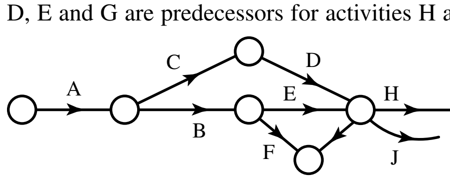
C is a predecessor for activity D.



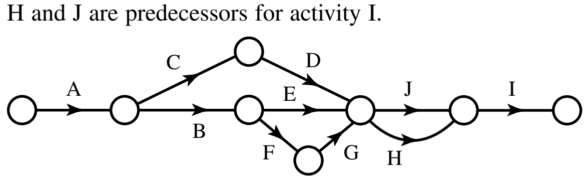
F is a predecessor for activity G.



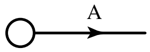
D, E and G are predecessors for activities H and J.

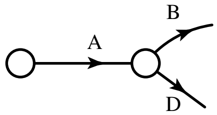


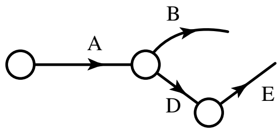
H and J are predecessors for activity I.



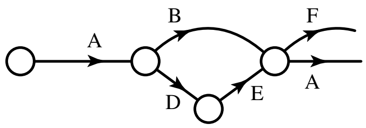
20 a First edge A has no predecessor
A is a predecessor for B and D.
D is a predecessor for E.



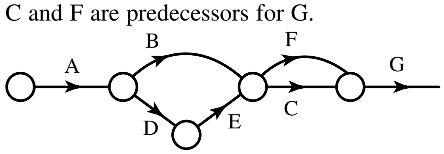




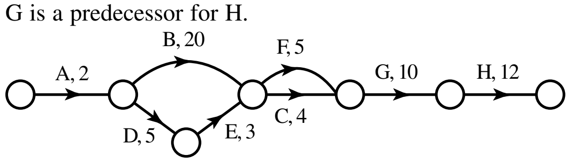
E is a predecessor for F.
B and E are predecessors for C.



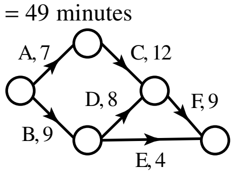
C and F are predecessors for G.



G is a predecessor for H.

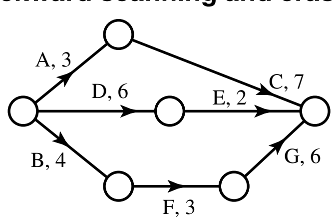


b Minimum time in which all tasks could be completed follow the path
A—B—F—G—H
= 2 + 20 + 5 + 10 + 12
= 49 minutes

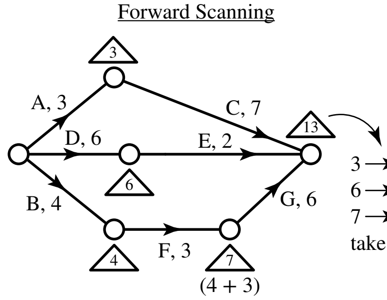


Exercise 10.3 — Critical path analysis with backward scanning and crashing

1

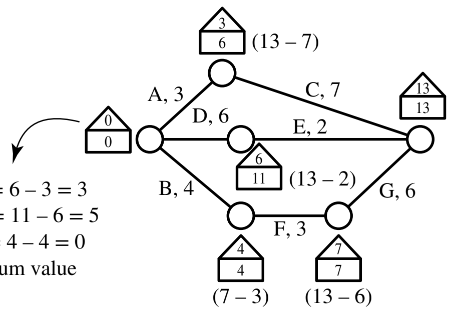


Forward Scanning



3 ⟶ C = 3 + 7 = 10
6 ⟶ E = 6 + 2 = 8
7 ⟶ G = 7 + 6 = 13*
take maximum value

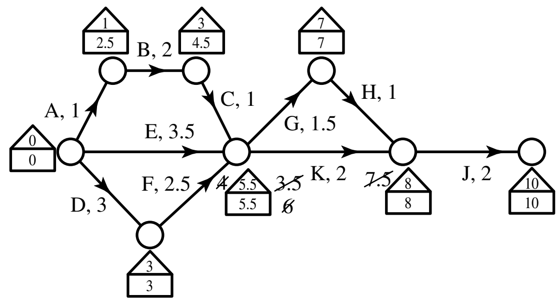
Backward Scanning



6 ⟶ A = 6 – 3 = 3
11 ⟶ D = 11 – 6 = 5
*4 ⟶ B = 4 – 4 = 0
take maximum value

Critical path = B—F—G
Float time for the non-critical activities
A: 3 hours
C: 3 hours
D: 5 hours
E: 5 hours

2 a Enter 10 for the last node.
Enter 8 for the next node before activity J.
Enter 7 for the node before activity H.
Enter either 8 − 2 (path J, K) or 8 − 1 − 1.5 (path H, G) for the node before activities G, K.
Reject the path J, K.
Enter 5.5 − 2.5 = 3 for the node before activity F.
Enter 5.5 − 1 = 4.5 for the node before activity C.
Enter 4.5 − 2 = 2.5 for the node before activity B.
Complete by entering a 0 at the start node.



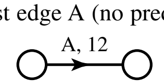
The critical path is where triangle number = box number:
D—F—G—H—J.

b Float times for non-critical activities:
Float(K) = 8 − 5.5 − 2 = 0.5 hours
Float(C) = 5.5 − 3 − 1 = 1.5 hours
Float(B) = 4.5 − 1 − 2 = 1.5 hours
Float(A) = 2.5 − 0 − 1 = 1.5 hours
Float(E) = 5.5 − 0 − 3.5 = 2 hours

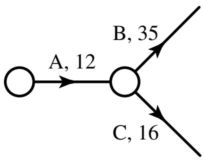
3 a

Activity letter	Immediate predecessor	Time
A	—	12
B	A	35
C	A	16
D	B	20
E	C	12
F	C	5
G	D, E	18

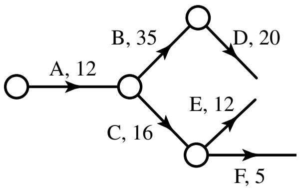
b First edge A (no predecessor).



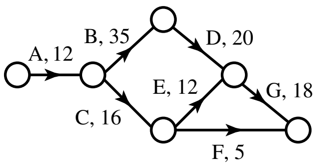
A is a predecessor for B and C.



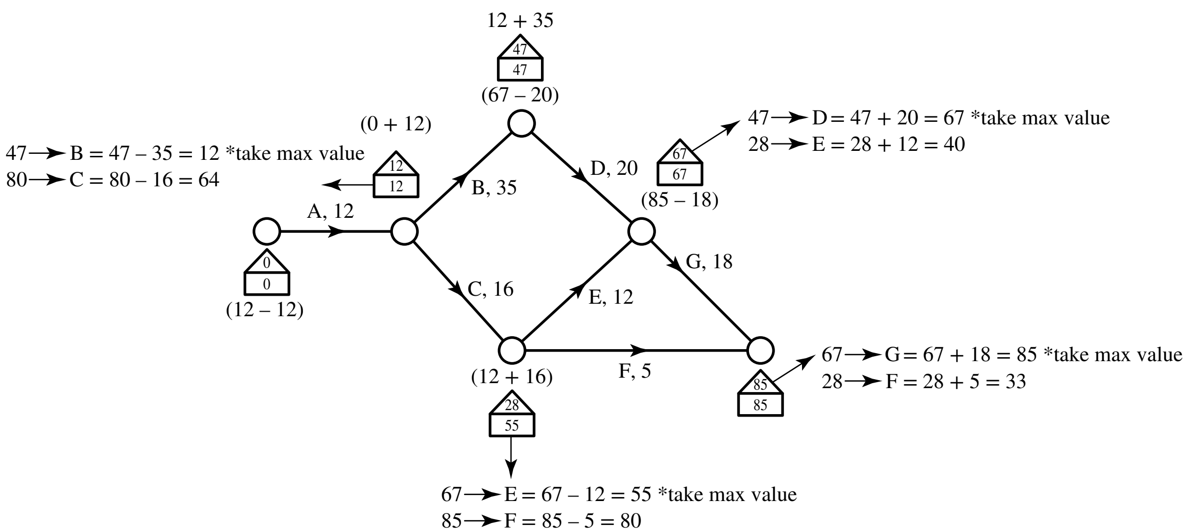
B is a predecessor for D.
C is a predecessor for E and F.



D and E are predecessors for G.



c

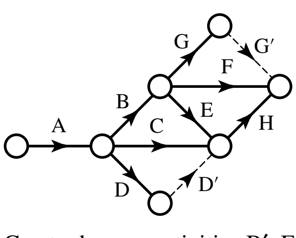


The earliest completion time is 85 minutes.

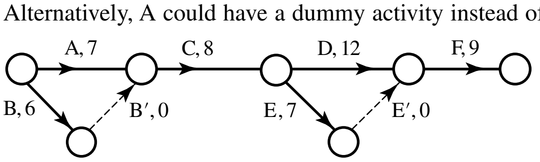
d Critical path = A—B—D—G

4 Activities with float time are not on the critical path. These activities would be C, E and F. The answer is **C**.

5



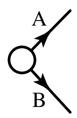
6 Create dummy activities B′, E′, since C has both A and B as immediate predecessors and F has both D and E as immediate predecessors.
Alternatively, A could have a dummy activity instead of B, and D could have a dummy activity instead of E.



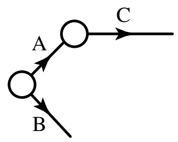
7

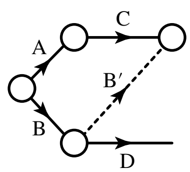
Activity letter	Immediate predecessor	Time (*h*)
A	—	3
B	—	5
C	A	7
D	B	7
E	B, C	1
F	D, E	2

A and B are first edges and have no predecessors.

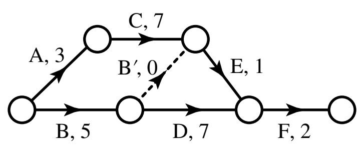


A is a predecessor of C.
B is a predecessor of D and E.
C also a predecessor of E.
(to skip a parallel edge use dummy edge B′)

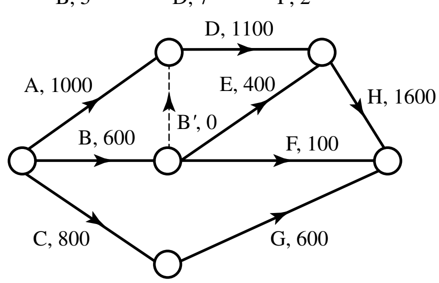




D and E are predecessors of F.

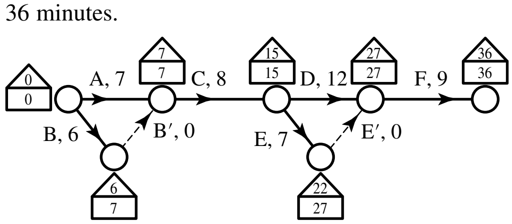


8



By examining outflow at A (1000 + 600 + 800) and inflow at F (100 + 600 + 1600), the maximum possible flow could be 2300.

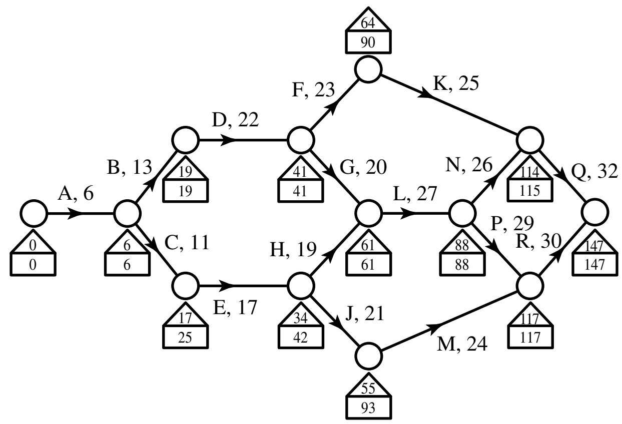
9 **a** Forward scan shows that the earliest completion time is 36 minutes.



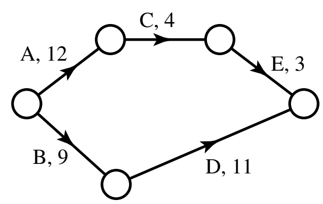
b Backward scan shows that the critical path is A—C—D—F.

c Float times for non-critical activities:
Float(E) = 27 − 15 − 7 = 5 minutes
Float(B) = 7 − 0 − 6 = 1 minute

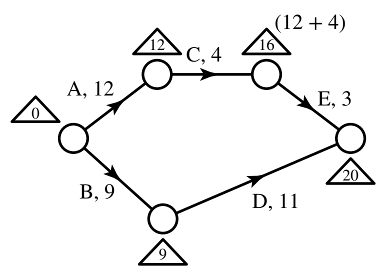
10 The critical path is A—B—D—G—L—P—R.



11

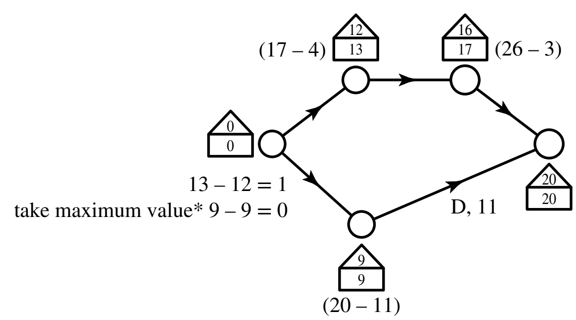


<u>Forward Scanning</u>



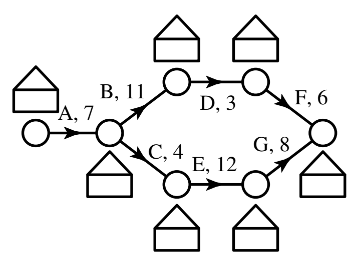
16 ⟶ E = 16 + 3 = 19
9 ⟶ D = 9 + 11 = 20
*take maximum value

<u>Backward Scanning</u>



Critical path is B—D
Float times for the non-critical activities
A:1 minute
C:1 minute
E:1 minute

12 **a**



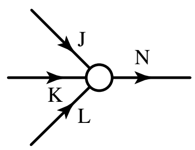
Earliest completion time = 31 days

b Critical path, from the network, where the 'triangle' numbers are equal to the 'box' numbers.
A—C—E—G

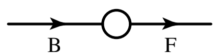
c If Activity E is reduced to 9 days, the A—C—E—G path is reduced to 28 days, which is still greater than the other possible path (A—B—D—F).

13 Float time for activity D
= 25 − 18 − 3
= 4 days
The answer is **D**.

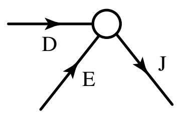
14 **a** J, K and L are predecessors of N.



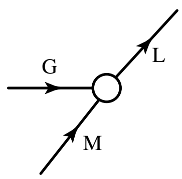
F is a predecessor of K.



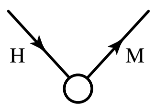
D and E are predecessors of J.



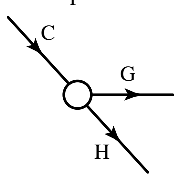
M and G are predecessors of L.



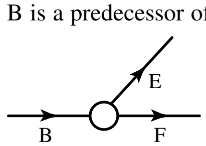
H is a predecessor of M.



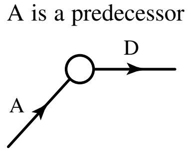
C is a predecessor of G and H.



B is a predecessor of E and F.



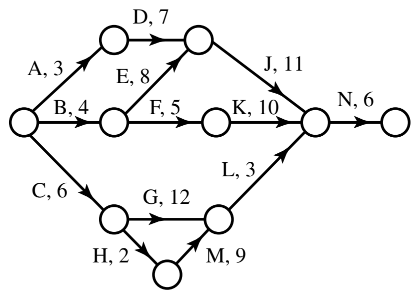
A is a predecessor of D.



A, B and C have no predecessors.

Activity letter	Immediate predecessor	Time
A	—	3
B	—	4
C	—	6
D	A	7
E	B	8
F	B	5
G	C	12
H	C	2
J	D, E	11
K	F	10
L	G, M	3
M	H	9
N	J, K, L	6

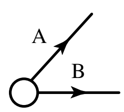
b and **c**



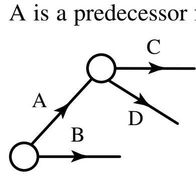
The earliest completion time is 29 minutes.
The critical path is B—E—J—N. (Where the 'triangle' numbers are equal to the 'box' numbers.)

d Non–critical activities are:
A, C, D, F, G, H, K, L and M.
Float times are
A:5 − 0 − 3 = 2
C:8 − 0 − 6 = 2
D:12 − 3 − 7 = 2
F:13 − 4 − 5 = 4
G:20 − 6 − 12 = 2
H:11 − 6 − 2 = 3
K:23 − 9 − 10 = 4
L:23 − 18 − 3 = 2
M:20 − 8 − 9 = 3

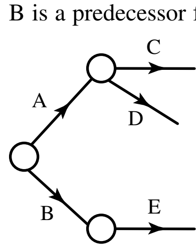
15 **a** A and B are first edges and have no predecessors.



A is a predecessor for C and D.

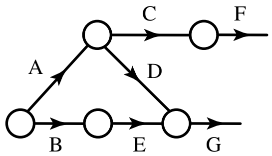


B is a predecessor for E.

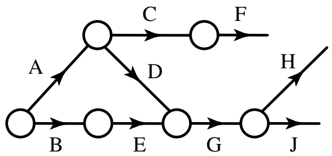


C is a predecessor for F.

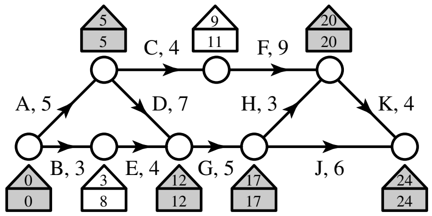
D and E are predecessors for G.



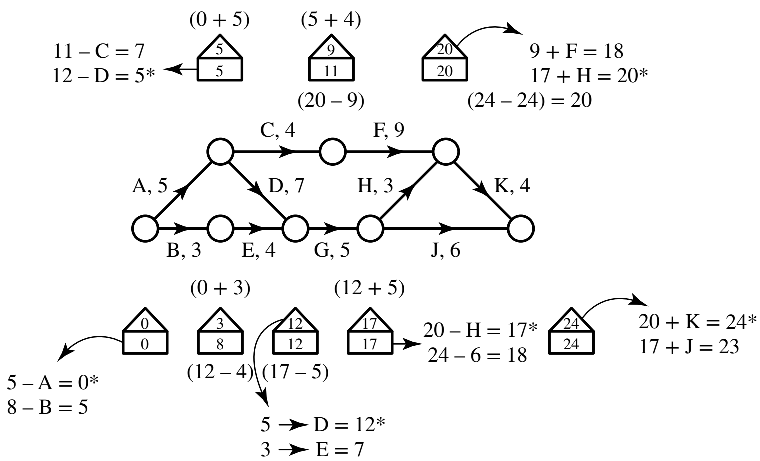
G is a predecessor for H and J.



F and H are predecessors for K.



b



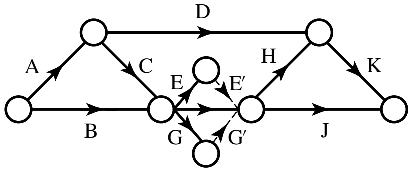
Earliest completion time = 24 hours.

c Critical path A—D—G—H—K

d

Activity	Time	EST	EFT	Float Time
B	3	0	8	5
C	4	5	11	2
E	4	3	12	5
F	9	9	20	2
J	6	17	24	1

16



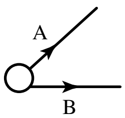
17 a

Activity letter	Immediate predecessor	Time (h)
A	—	11
B	—	9
C	A	2

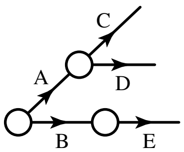
(*continued*)

Activity letter	Immediate predecessor	Time (*h*)
D	A	5
E	B	12
F	C	3
G	D	3
H	E	4
J	E, F, G	7

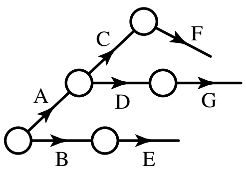
A and B are first edges and have no predecessors.



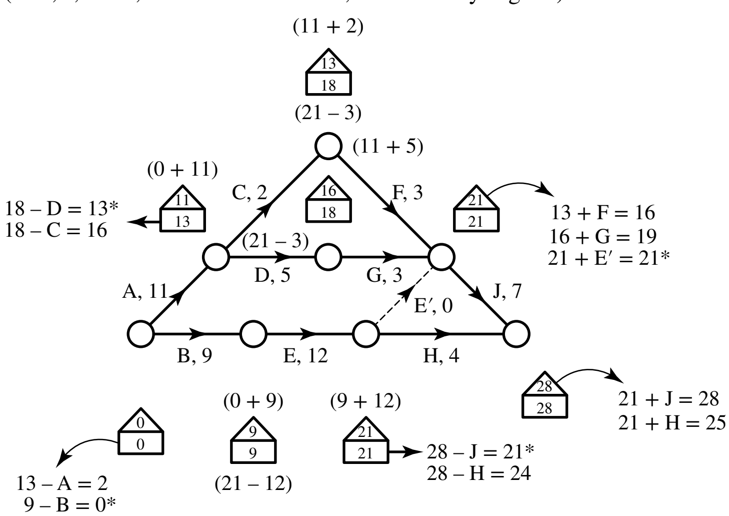
A is a predecessor of C and D.
B is a predecessor of E.



C is a predecessor of F.
D is a predecessor of G.



E is a predecessor of H and J.
F and G are predecessors of J as well
(as E, F, G—J, but E also needs —H, use a dummy edge E′).



Forward/Backward Scanning

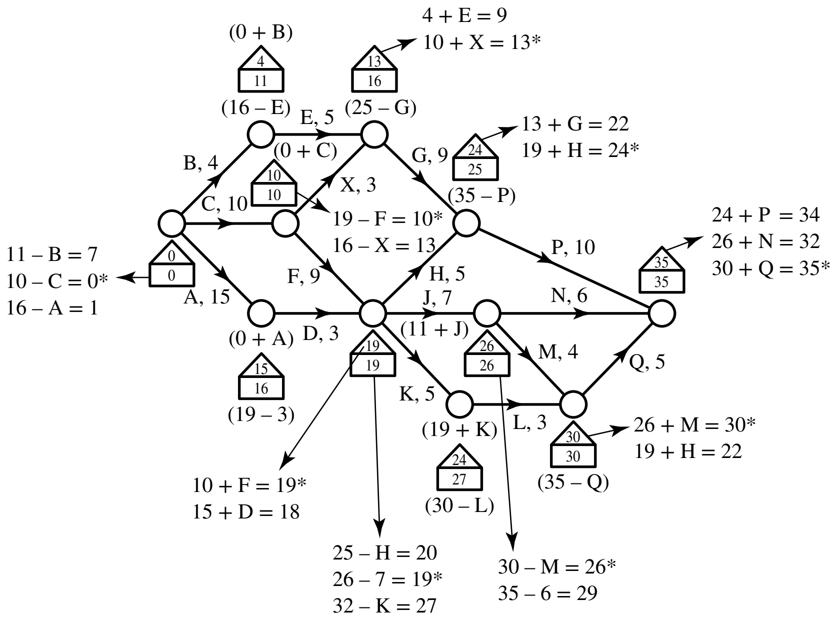
b Earliest completion time = 28 hours

c Critical path = B—E—E′—J

d

Activity	Time	EST	EFT	Float Time
A	11	0	13	2
C	2	11	18	5
D	5	11	18	2
F	3	13	21	5
G	3	16	21	2
H	4	21	28	3

18 **a** and **b**



Earliest completion time = 35 days

Critical path = C—F—J—M—Q

c Float time for activity X = EFT – EST – Time

= 16 – 10 – 3

= 3 days

d When J is reduced by 2 days to 5 days, the earliest completion time is reduced to 34 days. The new critical path becomes C—F—H—P. J is no longer a critical activity.

19 **a** LST for J = 29 − 8 = 21

So, LST for I = 17

So, activity time I = 24 − 17 = 7

LST for G = 24 − 4 = 20

LST for D′ = 17 − 0 = 17

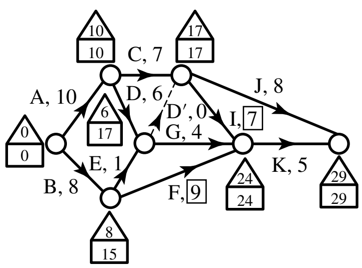
So, LFT for D and E = 17

LST for E = 17 − 1 = 16

So, LST for F = 15

So, activity time F = 24 − 15 = 9

LST for A and B = 0



b Critical path = A—C—I—K

c Float for F = 24 − 8 − 9 = 7

20 **a** Activity time for K = 44 − 38 = 6

LST for D = 25 − 9 = 16

LST for B′ = 16

LST for F = 38 − 16 = 22

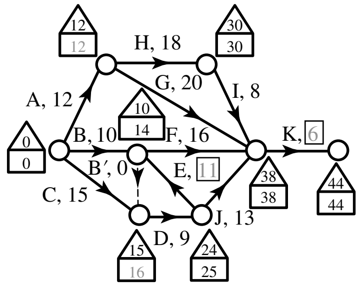
So, LST for E = 14

So, activity time E = 25 − 14 = 11

LST for H = 30 − 18 = 12

LST for G = 38 − 20 = 18

So, LFT for A = 12

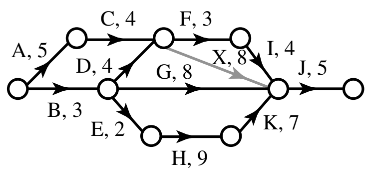


b Critical path = A—H—I—K

c Float time for F = 38 − 10 − 16 = 12

21 **a** Immediate predecessors of C and F are A, C and D respectively.
EST for G = 3 hours
EST for K = 3 + 2 + 9 = 14 hours

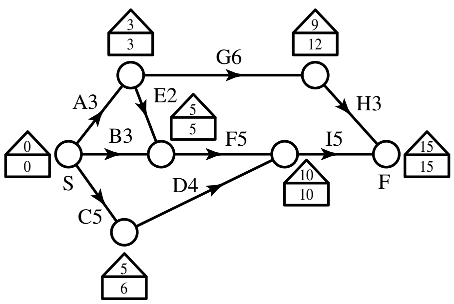
b EST for J = 17 and EST for F = 9
So, activity time X = 17 − 9 = 8 hours



c Critical path = A—C—X—J

d Earliest completion time = 22
LST for K = 22 − 7 = 15
LST for H = 15 − 9 = 6 hours after start

22



a 5 weeks

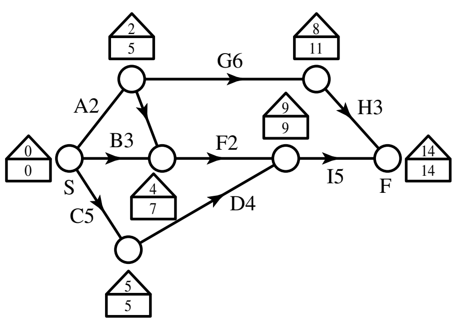
b Minimum time is 15 weeks.

c Critical path is A—E—F—I

d Slack time is 12 − 9 = 3 weeks

e Stages along the critical path can be shortened:
A—E—F—I (from the critical path)

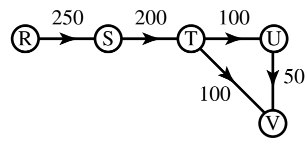
f After stages A and F being reduced to 2 weeks the new critical path will be C—D—I with a minimum time for completion 14 weeks.



Exercise 10.4 — Network flow

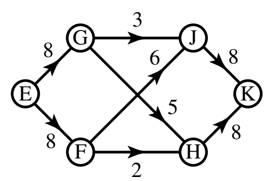
1

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	V	100
U	V	50

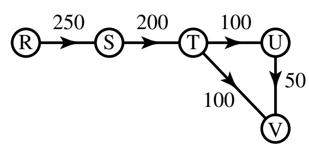


2

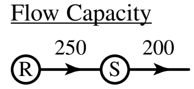
From	To	Flow capacity
E	F	8
E	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	K	8
H	K	8



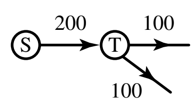
3



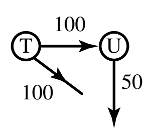
Flow Capacity



minimum = 200



minimum = 200

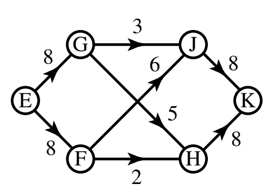


minimum = 150

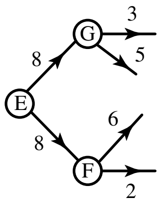
a ∴ Flow capacity = 150

b This does meet the demand as V requires 150.

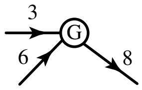
4



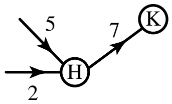
<u>Flow Capacity</u>



minimum = 16



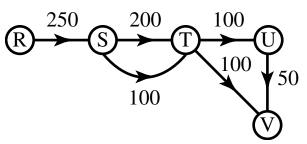
minimum = 15



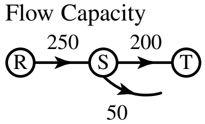
a ∴ Flow capacity = 15

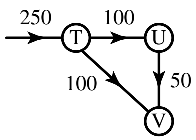
b No this does not meet the demand as K requires 16.

5 a

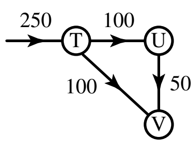


b Flow Capacity





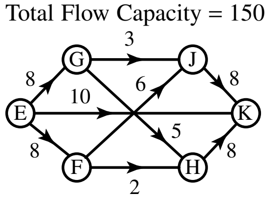
(Minimum = 50)



(Minimum = 100)

Total Flow Capacity = 150

6 a



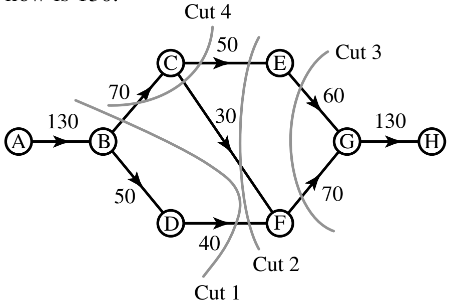
b Flow Capacity

Total Flow Capacity = 15 (as shown in question **2**)

+ 10 (E—K)

= 25

7 Considering outflow at A and inflow at H, the largest possible flow is 130.



Note that cut 4 is in fact <u>not</u> a cut as it fails to stop all flow.

Some possible cuts are:

CUT 1 = 70 + 40 = 110

CUT 2 = 50 + 30 + 40 = 120

CUT 3 = 60 + 70 = 130

Any other cut is more than 110.

Minimum cut = maximum flow = 110.

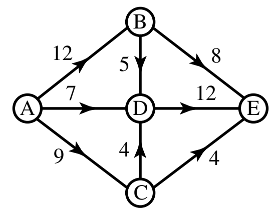
8 a Cut 2 is invalid as it does not stop all flow from A to E.

b CUT 1 = 9 + 7 + 12 = 28

CUT 3 = 4 + 4 + 7 + 5 + 8 = 28

CUT 4 = 4 + 4 + 7 + 12 = 27

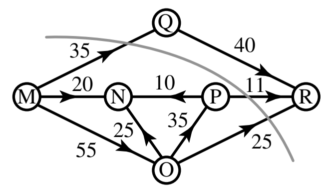
c



Minimum cut is 4 + 12 + 8 = 24

Therefore the maximum flow is 24.

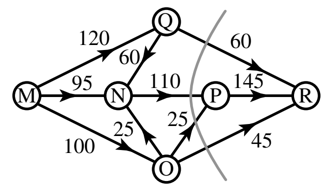
9 a



b Maximum flow = 71

(35 + 11 + 25)

10 a

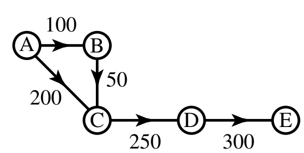


b Maximum flow = 240

(60 + 110 + 25 + 45)

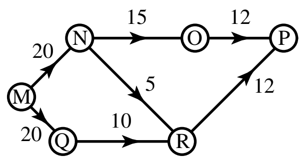
11 a

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300

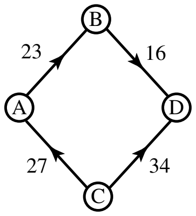


b

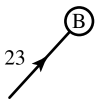
From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	P	12
R	P	12



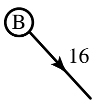
12



a The inflow of B is 23.

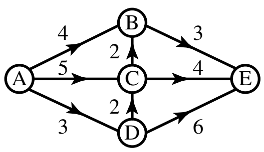


b Edge capacity flowing out of B is 16.

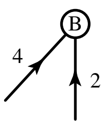


c The outflow from B is the minimum of **a** and **b**, so 16.

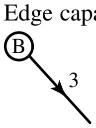
13



a The inflow of B is 4 + 2 = 6.

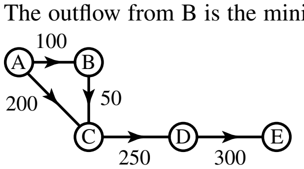


b Edge capacity flowing out of B is 3.



c The outflow from B is the minimum of **a** and **b**, so 3.

14 a



<u>Flow capacity</u>

outflow

B 50 } meets C demand
A 200 } of 250

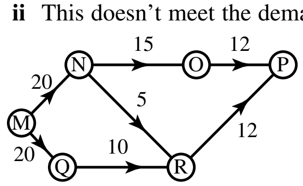
↑ ↑

C 250 → so D can only have 250

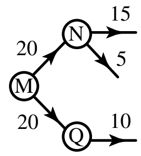
i ∴ Flow capacity = 250

ii This doesn't meet the demand as E requires 300.

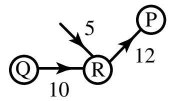
b



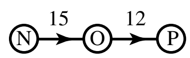
<u>Flow Capacity</u>



minimum = 30



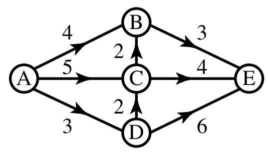
minimum = 24



i ∴ Flow capacity = 24

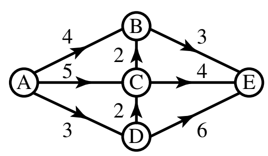
ii This does meet the demand as P requires 24.

15 a



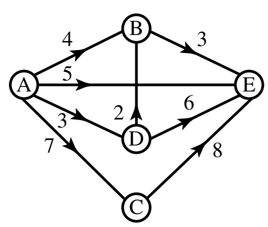
From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
C	B	2
C	E	4
D	C	2
D	E	6

b



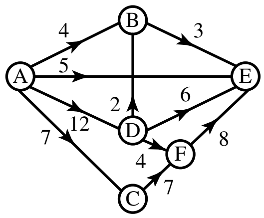
From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
B	C	2
C	E	4
D	C	2
D	E	6

c



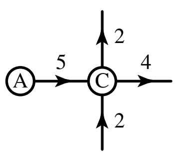
From	To	Flow capacity
A	B	4
A	C	7
A	D	3
A	E	5
B	E	3
C	E	8
D	B	2
D	E	6

d

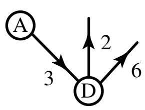


From	To	Flow capacity
A	B	4
A	C	7
A	D	12
A	E	5
C	F	7
D	B	2
D	E	6
D	F	4
F	E	8

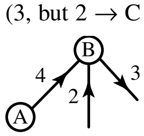
16 a



(5, but 2 → B ⇒ 4)



(3, but 2 → C ⇒ 1)



(3)

but A → B doesn't use all 6, so extra 2 from C not required and ∴ extra 2 from D not required

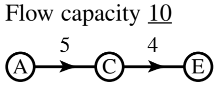
B → 3

C → 4

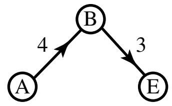
D → 3

Flow capacity <u>10</u>

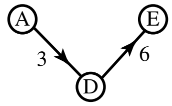
b



(minimum flow = 4) Capacity met on this flow, so flow from B and C not required.



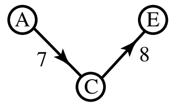
(Minimum flow = 3)



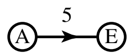
(Minimum flow = 3)

Total Flow Capacity = 4 + 3 + 3
= 10

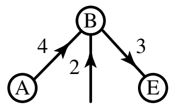
c



(Minimum = 7)

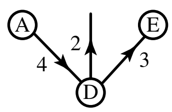


(Minimum = 5)



(Minimum = 3)

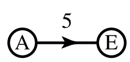
not required



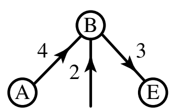
(Minimum = 3)

Total Flow Capacity = 18

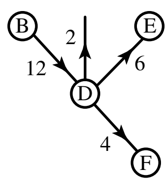
d



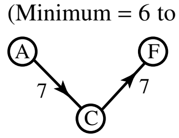
(Minimum = 5)

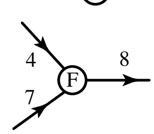


(Minimum = 3)



(Minimum = 6 to E and D)

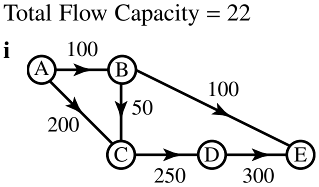




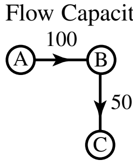
(Minimum = 8)

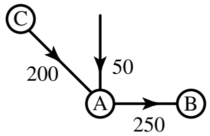
Total Flow Capacity = 22

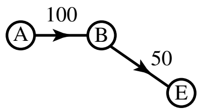
17 a i



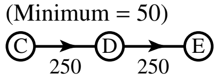
ii Flow Capacity





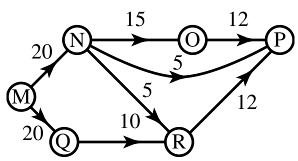


(Minimum = 50)

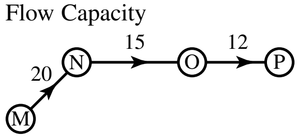


(Minimum = 250)
Total Flow Capacity = 300

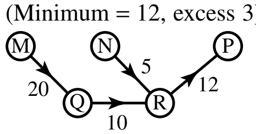
b i



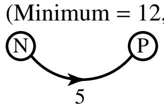
ii Flow Capacity



(Minimum = 12, excess 3)

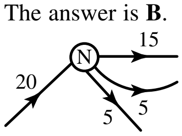


(Minimum = 12, excess 3)



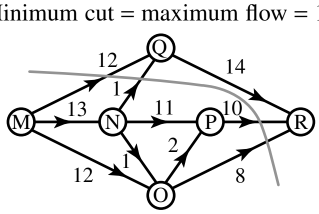
(Minimum = 5)
Total Flow Capacity = 29

18 In question **17b**, the outflow from N is 20.
The answer is **B**.



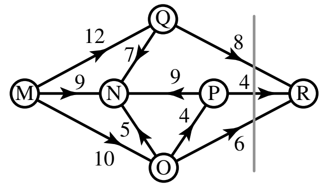
19 a Cut 4 is invalid since it does not stop all flow from A to G.
b Cut 1 = 6 + 9 + 10 = 25
Cut 2 = 5 + 8 + 13 = 26
Cut 3 = 11 + 13 = 24
Cut 5 = 6 + 8 + 5 + 10 = 29
c Minimum cut = maximum flow = 11 + 5 + 4 = 20

20 a i



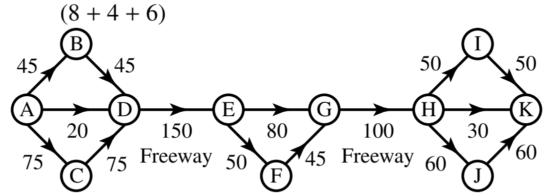
ii Maximum flow = 31
(12 + 1 + 10 + 8)

b i

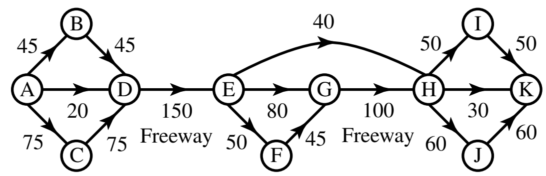


ii Maximum flow = 18
(8 + 4 + 6)

21



22 a There would be a traffic jam because inflow > outflow.
Node E can only handle (80 + 50) = 130.
b At node H, the traffic should flow smoothly as the inflow (100) is less than the capacity of flows leading from H.
(50 + 30 + 60 = 140)
c

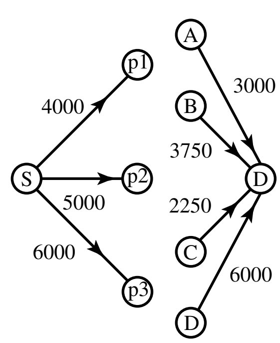


As node H has the potential to carry another 40 cars, then join a road between E and H.

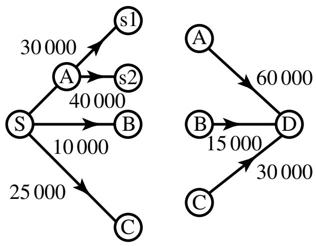
Exercise 10.5 — Assignment problems and bipartite graphs

1 Electricity produced = supply
4000 kWh, 5000 kWh,
and 6000 kWh
Total = 15,000 kWh
Towns supplied = demand

$$\begin{aligned}\text{Town A} &= 20\% \text{ of } 15\,000\\ &= 3000 \text{ kWh}\\ \text{Town B} &= 25\% \text{ of } 15\,000\\ &= 3750 \text{ kWh}\\ \text{Town C} &= 15\% \text{ of } 15\,000\\ &= 2250 \text{ kWh}\\ \text{Town D} &= 15\,000 - (3000 + 3750 + 2250)\\ &= 6000 \text{ kWh}\end{aligned}$$

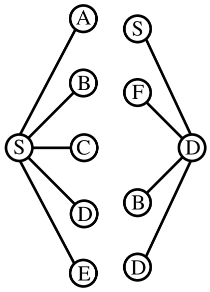


2 a



b Send 30 000 from S1 to A, 30 000 from S2 to A, 10 000 from S2 to B, 5000 from B to B, 5000 from B to C and 25 000 from C to C. (This may not be the cheapest method.)

3



4 Based on information in question 3, 'Brian and Chris between them have more different dishes than David and Earl'. The answer is **D**.
Brian and Chris have Fish, Soup, Beef and Dessert while David and Earl have Dessert, Beef and Fish.

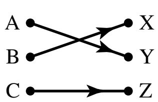
5

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 6 & 3 & 7 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{bmatrix} \end{array}$$

Subtract
Smallest in Row A = 3
in Row B = 2
in Row C = 2

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix} \end{array}$$

Number of lines = number of columns



There is only 1 possible allocation:
A → Y, B → X, C → Z
Total time = 3 + 2 + 2
= 7 hours

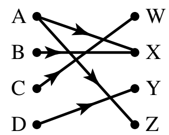
6

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 4 & 3 & 7 & 3 \\ 9 & 4 & 6 & 5 \\ 5 & 6 & 7 & 8 \\ 4 & 8 & 3 & 5 \end{bmatrix} \end{array}$$

Subtract
Smallest in Row A = 3
in Row B = 4
in Row C = 5
in Row D = 3

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 1 & 0 & 3 & 0 \\ 5 & 0 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 5 & 0 & 2 \end{bmatrix} \end{array}$$

Number of lines = number of columns



Allocation: D → Y, C → W, B → X, so A → Z
Total time = 3 + 4 + 5 + 3
= 15 hours

7

$$\begin{bmatrix} 6 & 9 & 9 & 4 \\ 10 & 9 & 9 & 7 \\ 4 & 9 & 6 & 3 \\ 5 & 8 & 8 & 6 \end{bmatrix}$$

Row reduction
Subtract smallest in Row 1 = 4
2 = 7
3 = 3
4 = 5

$$\begin{bmatrix} 2 & 5 & 5 & 0 \\ 3 & 2 & 2 & 0 \\ 1 & 6 & 3 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$$

Column reduction
Subtract smallest in Column 1 = 0
2 = 2
3 = 2
4 = 0

$$\begin{bmatrix} 2 & 3 & 3 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Smallest uncovered number = 1

$$\begin{bmatrix} 2 & 3 & 3 & 1 \\ 4 & 1 & 1 & 2 \\ 1 & 4 & 1 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix}$$

Subtract overall smallest number = 1

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

4 lines required
Minimum total allocation
4 + 6 + 9 + 5 = 24
Solved by Hungarian Algorithm

8

$$\begin{bmatrix} 5 & 23 & 14 & 9 \\ 11 & 29 & 6 & 14 \\ 21 & 17 & 14 & 13 \\ 20 & 27 & 22 & 8 \end{bmatrix}$$

Row reduction subtract
Smallest number in Row 1 = 5
2 = 6
3 = 13
4 = 8

$$\begin{bmatrix} 0 & 18 & 9 & 4 \\ 5 & 23 & 0 & 8 \\ 8 & 4 & 1 & 0 \\ 12 & 19 & 14 & 0 \end{bmatrix}$$

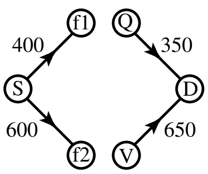
Column reduction subtract
Smallest number in Column 1 = 0
2 = 4
3 = 0
4 = 0

$$\begin{bmatrix} 0 & 14 & 9 & 4 \\ 5 & 19 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 2 & 15 & 14 & 0 \end{bmatrix}$$

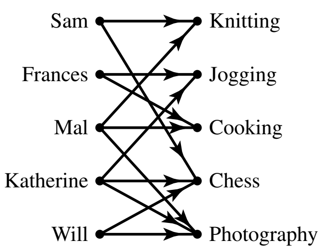
Solved by Column reduction.
Minimum total allocation $= 5+17+6+8$
$= 36$

9 Produces ∴ supply

- produces 1000 copies per month
  400 → Factory 1
  600 → Factory 2
  Distributed ∴ Demand
- Queensland = 350
  Victoria = 1000 − 350
  = 650



10 From the bipartite graph it can be said that:
"Mal and Frances, in total, have more hobbies than Sam and Will". The answer is **B**.

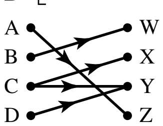


11

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 16 & 14 & 20 & 13 \\ 15 & 16 & 17 & 16 \\ 19 & 13 & 13 & 18 \\ 22 & 26 & 20 & 24 \end{bmatrix} \end{array}$$

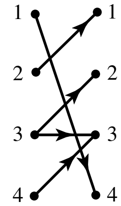
Subtract
Smallest in Row A = 13
in Row B = 15
in Row C = 13
in Row D = 20

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 3 & 1 & 7 & 0 \\ 0 & 1 & 2 & 1 \\ 6 & 0 & 0 & 5 \\ 2 & 6 & 0 & 4 \end{bmatrix} \end{array}$$



A → Z, B → W
C → X (as no other to X)
∴ D → Y
Total time 15 + 13 + 13 + 20 = 61 hours

12



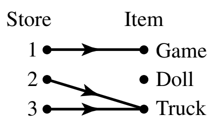
13

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{Game} & \text{Doll} & \text{Truck} \end{array} \\ \begin{bmatrix} 30 & 50 & 35 \\ 45 & 50 & 30 \\ 40 & 60 & 30 \end{bmatrix} \end{array}$$

Subtract
Smallest in Row 1 = 30
Row 2 = 30
Row 3 = 30

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{Game} & \text{Doll} & \text{Truck} \end{array} \\ \begin{bmatrix} 0 & 20 & 5 \\ 15 & 20 & 0 \\ 10 & 30 & 0 \end{bmatrix} \end{array}$$

Minimum cost
1 → G = $30
Between Store 2 and 3 it is cheaper to buy the Doll from Store 2 ∴ 3 → T



∴ 30 + 50 + 30 = $110

14

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 7 & 3 & 7 \\ 3 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix}$$

Subtract
Smallest in Row A = 3
in Row B = 3
in Row C = 5

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 4 & 0 & 4 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Allocation. A → 3, B → 3, C → 5
∴ 3 + 3 + 5 = 11
The answer is **B**.

15 a

$$\begin{array}{r} \\ \text{Car 1} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} D1 & D2 & D3 & D4 & D5 \end{array} \\ \begin{bmatrix} 20 & 15 & 17 & 16 & 18 \\ 17 & 15 & 19 & 17 & 16 \\ 18 & 19 & 16 & 19 & 16 \\ 19 & 19 & 17 & 21 & 17 \\ 24 & 19 & 17 & 17 & 17 \end{bmatrix} \end{array}$$

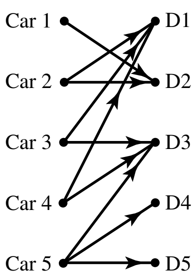
Row Reduction
Subtract
Smallest in Row 1 = 15
2 = 15
3 = 16
4 = 17
5 = 17

$$\begin{array}{c} \\ \text{Car1} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} \text{D1} & \text{D2} & \text{D3} & \text{D4} & \text{D1} \end{array} \\ \begin{bmatrix} 5 & 0 & 2 & 1 & 3 \\ 2 & 0 & 4 & 2 & 1 \\ 2 & 3 & 0 & 3 & 0 \\ 2 & 2 & 0 & 4 & 0 \\ 7 & 2 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Column Reduction
Subtract
Smallest in Column 1 = 2
2 = 0
3 = 0
4 = 0
5 = 0

$$\begin{array}{c} \\ \text{Car1} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} \text{D1} & \text{D2} & \text{D3} & \text{D4} & \text{D1} \end{array} \\ \begin{bmatrix} 3 & 0 & 2 & 1 & 3 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 5 & 2 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b



c **i** C1 → D2, C2 → D1, C5 → D4, C3 → D3, C4 → D5 or C1 → D2, C2 → D1, C5 → D4, C3 → D5, C4 → D3

ii Total = $82 000

16 **a**

$$\begin{bmatrix} 10 & 15 & 12 & 17 \\ 17 & 21 & 19 & 14 \\ 16 & 22 & 17 & 19 \\ 23 & 26 & 29 & 27 \end{bmatrix}$$

Row reduction
Subtract smallest in Row 1 = 10
2 = 14
3 = 16
4 = 23

$$\begin{bmatrix} 0 & 5 & 2 & 7 \\ 3 & 7 & 5 & 0 \\ 0 & 6 & 1 & 3 \\ 0 & 3 & 6 & 4 \end{bmatrix}$$

Column Reduction
Subtract smallest in Column 1 = 0
2 = 3
3 = 1
4 = 0

$$\begin{bmatrix} 0 & 2 & 1 & 7 \\ 3 & 4 & 4 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 5 & 4 \end{bmatrix}$$

Minimum number of lines required = 4
Solved at Column reduction.
Minimum total allocation = 10 + 26 + 17 + 14
= 67

b

$$\begin{bmatrix} 12 & 10 & 11 & 13 & 11 \\ 11 & 11 & 13 & 12 & 12 \\ 12 & 16 & 13 & 16 & 12 \\ 9 & 10 & 9 & 11 & 9 \\ 14 & 11 & 11 & 11 & 11 \end{bmatrix}$$

Row reduction subtract
Smallest number in Row 1 = 10
2 = 11
3 = 12
4 = 9
5 = 11

$$\begin{bmatrix} 2 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 1 & 4 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solved by row reduction
Minimum total allocation = 10 + 11 + 12 + 9 + 11
= 53

17 **a**

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}\begin{array}{c} \begin{array}{cccc} \text{J1} & \text{J2} & \text{J3} & \text{J4} \end{array} \\ \begin{bmatrix} 30 & 40 & 50 & 60 \\ 70 & 30 & 40 & 70 \\ 60 & 50 & 60 & 30 \\ 20 & 80 & 50 & 70 \end{bmatrix} \end{array}$$

Row Reduction
Subtract
Smallest number in Row A = 30
B = 30
C = 30
D = 20

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}\begin{array}{c} \begin{array}{cccc} \text{J1} & \text{J2} & \text{J3} & \text{J4} \end{array} \\ \begin{bmatrix} 0 & 10 & 20 & 30 \\ 40 & 0 & 10 & 40 \\ 30 & 20 & 30 & 0 \\ 0 & 60 & 30 & 50 \end{bmatrix} \end{array}$$

Column Reduction
Subtract
Smallest number in Column J1 = 0
J2 = 0
J3 = 10
J4 = 0

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}\begin{array}{c} \begin{array}{cccc} \text{J1} & \text{J2} & \text{J3} & \text{J4} \end{array} \\ \begin{bmatrix} 0 & 10 & 10 & 30 \\ 40 & 0 & 0 & 40 \\ 30 & 20 & 20 & 0 \\ 0 & 60 & 20 & 50 \end{bmatrix} \end{array}$$

b Hungarian Algorithm
Smallest uncovered number = 10

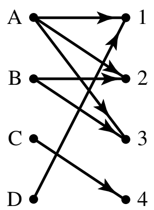
$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}\begin{array}{c} \begin{array}{cccc} \text{J1} & \text{J2} & \text{J3} & \text{J4} \end{array} \\ \begin{bmatrix} 10 & 10 & 10 & 30 \\ 60 & 10 & 10 & 50 \\ 50 & 30 & 30 & 10 \\ 10 & 60 & 20 & 50 \end{bmatrix} \end{array}$$

Overall smallest number = 10

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}\begin{array}{c} \begin{array}{cccc} \text{J1} & \text{J2} & \text{J3} & \text{J4} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 20 \\ 50 & 0 & 0 & 40 \\ 40 & 20 & 20 & 0 \\ 0 & 50 & 10 & 40 \end{bmatrix} \end{array}$$

(One possible result)

c



d **i** A → 2, B → 3, C → 4, D → 1 or A → 3, B → 2, C → 4, D → 1

ii Total = 40 + 40 + 30 + 20
= 130 minutes

18

$$\begin{array}{cc} & \begin{matrix} J1 & J2 & J3 & J4 \end{matrix} \\ \begin{matrix} T \\ U \\ V \\ W \end{matrix} & \begin{bmatrix} 100 & 50 & 35 & 55 \\ 60 & 45 & 70 & 55 \\ 40 & 70 & 50 & 30 \\ 70 & 50 & 70 & 70 \end{bmatrix} \end{array}$$

Row Reduction
Subtract Smallest in Row T = 35
U = 45
V = 30
W = 50

$$\begin{array}{cc} & \begin{matrix} J1 & J2 & J3 & J4 \end{matrix} \\ \begin{matrix} T \\ U \\ V \\ W \end{matrix} & \begin{bmatrix} 65 & 15 & 0 & 20 \\ 15 & 0 & 25 & 10 \\ 10 & 40 & 20 & 0 \\ 20 & 0 & 20 & 20 \end{bmatrix} \end{array}$$

Column Reduction
Subtract smallest number from Column 1 = 10
2 = 0
3 = 0
4 = 0

$$\begin{array}{cc} & \begin{matrix} J1 & J2 & J3 & J4 \end{matrix} \\ \begin{matrix} T \\ U \\ V \\ W \end{matrix} & \begin{bmatrix} 55 & 15 & 0 & 20 \\ 5 & 0 & 25 & 10 \\ 0 & 40 & 20 & 0 \\ 10 & 0 & 20 & 20 \end{bmatrix} \end{array}$$

Hungarian Algorithm
Smallest uncovered number = 5

$$\begin{array}{cc} & \begin{matrix} J1 & J2 & J3 & J4 \end{matrix} \\ \begin{matrix} T \\ U \\ V \\ W \end{matrix} & \begin{bmatrix} 60 & 25 & 5 & 25 \\ 5 & 5 & 25 & 10 \\ 5 & 50 & 25 & 5 \\ 10 & 5 & 20 & 20 \end{bmatrix} \end{array}$$

Smallest number = 5

$$\begin{array}{cc} & \begin{matrix} J1 & J2 & J3 & J4 \end{matrix} \\ \begin{matrix} T \\ U \\ V \\ W \end{matrix} & \begin{bmatrix} 55 & 20 & 0 & 0 \\ 0 & 0 & 20 & 5 \\ 0 & 45 & 20 & 0 \\ 5 & 0 & 15 & 5 \end{bmatrix} \end{array}$$

a T → 3, U → 1, V → 4, W → 2

b Time = 35 + 60 + 30 + 50
= 175 minutes

19 **a**

$$\begin{array}{cc} & \begin{matrix} A & C & F & G \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 60 & 78 & 67 & 37 \\ 45 & 80 & 70 & 90 \\ 60 & 35 & 70 & 86 \\ 42 & 66 & 54 & 72 \end{bmatrix} \end{array}$$

First modify the minimisation problem, by subtracting each number by the overall largest value, 90.

$$\begin{array}{cc} & \begin{matrix} A & C & F & G \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 30 & 12 & 23 & 53 \\ 45 & 10 & 20 & 0 \\ 30 & 55 & 20 & 4 \\ 48 & 24 & 36 & 18 \end{bmatrix} \end{array}$$

Row Reduction subtract
Smallest from row K = 12
L = 0
M = 4
N = 18

$$\begin{array}{cc} & \begin{matrix} A & C & F & G \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 18 & 0 & 11 & 41 \\ 45 & 10 & 20 & 0 \\ 26 & 51 & 16 & 0 \\ 30 & 6 & 18 & 0 \end{bmatrix} \end{array}$$

Column Reduction subtract
Smallest from column A = 18
C = 0
F = 11
G = 0

b

$$\begin{array}{cc} & \begin{matrix} A & C & F & G \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 41 \\ 27 & 10 & 9 & 0 \\ 8 & 51 & 5 & 0 \\ 12 & 6 & 7 & 0 \end{bmatrix} \end{array}$$

c To solve, continue with Hungarian Algorithm
Smallest uncovered number = 5

$$\begin{array}{cc} & \begin{matrix} A & C & F & G \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 5 & 5 & 5 & 51 \\ 27 & 10 & 9 & 5 \\ 8 & 51 & 5 & 5 \\ 12 & 6 & 7 & 5 \end{bmatrix} \end{array}$$

Subtract smallest number from all = 5

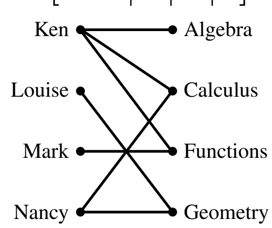
$$\begin{array}{cc} & \begin{matrix} A & C & F & G \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 46 \\ 22 & 5 & 4 & 0 \\ 3 & 46 & 0 & 0 \\ 7 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Repeat
Smallest uncovered number = 1

$$\begin{array}{cc} & \begin{matrix} A & C & F & G \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 48 \\ 22 & 5 & 4 & 1 \\ 4 & 47 & 1 & 2 \\ 7 & 1 & 2 & 1 \end{bmatrix} \end{array}$$

Subtract smallest number from all = 1

$$\begin{array}{cc} & \begin{matrix} A & C & F & G \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 47 \\ 21 & 4 & 3 & 0 \\ 3 & 46 & 0 & 1 \\ 6 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



Ken → Algebra,
Louise → Geometry,

Mark → Functions,
Nancy → Calculus

d Average score

$= \frac{60+90+70+66}{4}$

$= 71.5\%$

20

$$\begin{array}{cc} & \begin{array}{cccc} \text{P1} & \text{P2} & \text{P3} & \text{P4} \end{array} \\ \begin{array}{c} \text{V1} \\ \text{V2} \\ \text{V3} \\ \text{V4} \end{array} & \begin{bmatrix} 13 & 17 & 14 & 23 \\ 8 & 12 & 17 & 9 \\ 9 & 17 & 14 & 11 \\ 21 & 16 & 13 & 14 \end{bmatrix} \end{array}$$

Row Reduction subtract
Smallest number from Row 1 = 13
2 = 8
3 = 9
4 = 13

$$\begin{array}{cc} & \begin{array}{cccc} \text{P1} & \text{P2} & \text{P3} & \text{P4} \end{array} \\ \begin{array}{c} \text{V1} \\ \text{V2} \\ \text{V3} \\ \text{V4} \end{array} & \begin{bmatrix} 0 & 4 & 1 & 10 \\ 0 & 4 & 9 & 1 \\ 0 & 8 & 5 & 2 \\ 8 & 3 & 0 & 1 \end{bmatrix} \end{array}$$

Column Reduction subtract
Smallest number from Column 1 = 0
2 = 3
3 = 0
4 = 1

$$\begin{array}{cc} & \begin{array}{cccc} \text{P1} & \text{P2} & \text{P3} & \text{P4} \end{array} \\ \begin{array}{c} \text{V1} \\ \text{V2} \\ \text{V3} \\ \text{V4} \end{array} & \begin{bmatrix} 0 & 1 & 1 & 9 \\ 0 & 1 & 9 & 0 \\ 0 & 5 & 5 & 1 \\ 8 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Smallest uncovered number = 1

$$\begin{array}{cc} & \begin{array}{cccc} \text{P1} & \text{P2} & \text{P3} & \text{P4} \end{array} \\ \begin{array}{c} \text{V1} \\ \text{V2} \\ \text{V3} \\ \text{V4} \end{array} & \begin{bmatrix} 1 & 1 & 1 & 10 \\ 1 & 1 & 9 & 1 \\ 1 & 5 & 5 & 2 \\ 10 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

Subtract Smallest number = 1

$$\begin{array}{cc} & \begin{array}{cccc} \text{P1} & \text{P2} & \text{P3} & \text{P4} \end{array} \\ \begin{array}{c} \text{V1} \\ \text{V2} \\ \text{V3} \\ \text{V4} \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 4 & 4 & 1 \\ 9 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

a V1 → P2, V2 → P4, V3 → P1, V4 → P3 or
V1 → P3, V2 → P4, V3 → P1, V4 → P2

b Total = 17 + 9 + 9 + 13
= 48 km