Point Cook Senior

Year 11

General Mathematics 2017

Strand: Algebra and Structure

Topic 1 – Linear relations and equations

In this area of study students cover representation and manipulation of linear relations and equations, including simultaneous linear equations, and their applications in a range of contexts.

This topic includes:

- substitution into, and transposition of linear relations, such as scale conversion
- solution of linear equations, including literal linear equations
- developing formulas from word descriptions and substitution of values into formulas and evaluation
- construction of tables of values from a given formula
- linear relations defined recursively and simple applications of these relations
- numerical, graphical and algebraic solutions of simultaneous linear equations in two variables
- use of linear equations, including simultaneous linear equations in two variables, and their application to solve practical problems.

Topic 10 – Graphs of linear and non-linear relations

In this area study students cover continuous models involving linear and non-linear relations and their graphs, linear inequalities and programming, and variation.

Linear graphs and models

This topic includes:

- review of linear functions and graphs
- the concept of a linear model and its specification

- the construction of a linear model to represent a practical situation including domain of application
- the interpretation of the parameters of a linear model and its use to make predictions, including the issues of
interpolation and extrapolation
- fitting a linear model to data by using the equation of a line fitted by eye
- use of piecewise linear (line segment) graphs to model and analyse practical situations.

Inequalities and linear programming

This topic includes:
- linear inequalities in one and two variables and their graphical representation
- the linear programming problem and its purpose
- the concepts of feasible region, constraint and objective function in the context of solving a linear programming
problem
- use of the corner-point principle to determine the optimal solution/s of a linear programming problem
- formulation and graphical solution of linear programming problems involving two variables.

Variation

This topic includes:
- numerical, graphical and algebraic approaches to direct, inverse and joint variation
- transformation of data to linearity to establish relationships between variables, for example y and x^2 , or y and 1/x
- modelling of given non-linear data using the relationships y = kx^2 + c and y = k/x + c where k > 0
- modelling of data using the logarithmic function y = a log10(x) + c where a > 0.

Chapter Sections	**Questions to be completed**
1.2 Linear relations	1,2,3,4,5,6,13,14
1.3 Solving linear equations	1,2,3,4,5,6,7,8,11,14,18
1.4 Developing linear equations	1,2,3,4,5,7,8,9,13,19,
10.2 Linear functions and graphs	1,2,3,4,5,10,11,23,24
10.3 Linear modelling	1,2,3,,4,5,6,7,8,17
10.4 Linear equations and predictions	1,2,3,4,5,7,8,10,17,18

CHAPTER 1 - LINEAR RELATIONS AND EQUATIONS

1.2 Linear relations

Identifying linear relations

When a linear relation is expressed as an equation, the highest power of both variables in the equation is 1.

EXAMPLE

Identify which of the following equations are linear.

(a) $y = 4x + 1$

Both x and y have a power of 1 so linear.

(b) $b = c^2 - 5c + 6$

c has a power of 2 so NOT!

(c) $y = \sqrt{x}$

x is square root or power of $\frac{1}{2}$ so NOT!

(d) $m^2 = 6(n - 10)$

m has a power of 2 so NOT!

(e) $d = \frac{3t+8}{7}$

d and t have a power of 1 so linear

(f) $y = 5^x$

x is the power so NOT!

Rules for linear relations

Rules define or describe relationships between two or more variables. Rules for linear relations can be found by determining the common difference between consecutive terms of the pattern formed by the rule.

EXAMPLE

Find the equations for the linear relations formed by the following number patterns.

Let n be the term number and t_n be the term value.

(a) 3,7,11,15

common difference = 4 so 4n

4 × 1 = 4 − 1 = 3

4 × 2 = 8 − 1 = 7

etc.

so $t = 4n - 1$

~~$y = 4n - 1$~~

(b) 8,5,2,−1 difference = −3

−3n

−3 × 1 = −3 + 11 = 8

−3 × 2 = −6 + 11 = 5

etc

so $t = -3n + 11$

Transposing linear equations

If we are given a linear equation between two variables, we are able to transpose this relationship. That is, we can change the equation so that the variable on the right-hand side of the equation becomes the stand-alone variable on the left-hand side of the equation.

EXAMPLE

Transpose the linear equation $y = 4x + 7$ to make x the subject of the equation.

$$y = 4x + 7$$
$$-7 \qquad\qquad -7$$

$$\frac{y-7}{4} = \frac{4x}{4}$$

$$\frac{y-7}{4} = x$$

1.3 Solving linear equations

Solving linear equations with one variable

To solve linear equations with one variable, all operations performed on the variable need to be identified in order, and then the opposite operations need to be performed in reverse order.

Solving

- Brackets ( )
- Order/indices ^
- Division $\div$
- Multiplication $\times$
- Addition $+$
- Subtraction $-$

Evaluating

EXAMPLE

Solve the following linear equations to find the unknowns.

(a) $5x = 12$

$$\frac{5x}{5} = \frac{12}{5}$$

$$x = \frac{12}{5}$$

(b) $8t + 11 = 20$

$$8t + 11_{-11} = 20_{-11}$$

$$8t = 20 - 11$$

$$\frac{8t}{8} = \frac{9}{8}$$

$$t = \frac{9}{8}$$

(c) $12 = 4(n-3)$

$$12 = 4n - 12$$

$$12^{+12} = 4n - 12^{+12}$$

$$\frac{24}{4} = \frac{4n}{4}$$

$$6 = n$$

(d) $\frac{4x-2}{3} = 5$

$$\frac{4x-2}{3} \times 3 = 5^{\times 3}$$

$$4x - 2^{+2} = 15^{+2}$$

$$\frac{4x}{4} = \frac{17}{4}$$

$$x = \frac{17}{4}$$

Substituting into linear equations

If we are given a linear equation between two variables and we are given the value of one of the variables, we can substitute this into the equation to determine the other value.

EXAMPLE

Substitute $x = 3$ into the linear equation $y = 2x + 5$ to determine the value of y.

$y = 2 \times 3 + 5$

$y = 6 + 5$

$y = 11$

Literal linear equations

A literal equation is an equation that includes several pronumerals or variables.

EXAMPLE

Solve the linear literal equation $y = mx + c$ for x.

1.4 Developing linear equations

Developing linear equations from word descriptions

To write a worded statement as a linear equation, we must first identify the unknown and choose a pronumeral to represent it. We can then use the information given in the statement to write a linear equation in terms of the pronumeral.

The linear equation can then be solved as before, and we can use the result to answer the original question.

EXAMPLE

Cans of soft drinks are sold at SupaSave in packs of 12 costing $5.40. Form and solve a linear equation to determine the price of 1 can of soft drink.

$$12s = 5.40$$

$$\frac{12s}{12} = \frac{5.40}{12}$$

$$s = 4.5$$

$= 45$ cents per can

Tables of values

Tables of values can be generated from formulas by entering given values of one variable into the formula.

EXAMPLE

The amount of water that is filling a tank is found by the rule $W = 100t + 20$, where W is the amount of water in the tank in litres and t is the time in hours.

(a) Generate a table of values that shows the amount of water, W, in the tank every hour for the first 8 hours (i.e. t=0,1,2,3,...,8).

When $t = 0$: $W = 100t + 20 = 100 \times 0 + 20 = 20$

When $t = 1$: $W = 100 \times 1 + 20 = 100 + 20 = 120$

When $t = 2$: $W = 100 \times 2 + 20 = 200 + 20 = 220$

When $t = 3$: $W = 100 \times 3 + 20 = 320$

When $t = 4$: $= 420$

When $t = 5$: $= 520$

When $t = 6$: $= 620$

When $t = 7$: $= 720$

When $t = 8$: $= 820$

t	0	1	2	3	4	5	6	7	8
W	20	120	220	320	420	520	620	720	820

(b) Using your table, how long in hours will it take for there to be over 700 litres in the tank?

It will take 7 hrs to get over 700 Litres.

Linear relations defined recursively

Many sequences of numbers are obtained by following rules that define a relationship between any one term and the previous term. Such a relationship is known as a recurrence relation.

A term in such a sequence is defined as t_n with n denoting the place in the sequence. The term t_n is the current term in the sequence, while the term t_{n+1} is the next term in the sequence.

If a recurrence relation is of a linear nature (that is, there is a common difference d between each term in the sequence) then we can define the recurrence relation as:

$$t_{n+1} = t_n + d, t_1 = a$$

This means that the first term in the sequence is a, and each subsequent (next) term is found by adding d to the current term.

EXAMPLE

A linear recurrence relation is given by the formula $t_{n+1} = t_n + 6, t_1 = 5$. Write the first six terms of the sequence.

When $n = 1$: 5

When $n = 2$: 11 = 5 + 6

When $n = 3$: 17 = 11 + 6

When $n = 4$: 23 = 17 + 6

When $n = 5$: 29 = 23 + 6

When $n = 6$: 35 = 29 + 6

The first six values are: 5, 11, 17, 23, 29, 35

EXAMPLE

The weekly rent on an inner-city apartment increases by \$10 every year. In a certain year the weekly rent is \$310.

(a) Model this situation by setting up a linear recurrence relation between the weekly rental prices in consecutive years.

$a = 310 \quad d = 10$

$t_n = t_{n-1} + 10 \quad . \; t_1 = 310$

(b) Find the weekly rent for the first six years.

When $n = 1$: 310

When $n = 2$: $310 + 10 = 320$

When $n = 3$: $320 + 10 = 330$

When $n = 4$: $330 + 10 = 340$

When $n = 5$: $340 + 10 = 350$

When $n = 6$: $350 + 10 = 360$

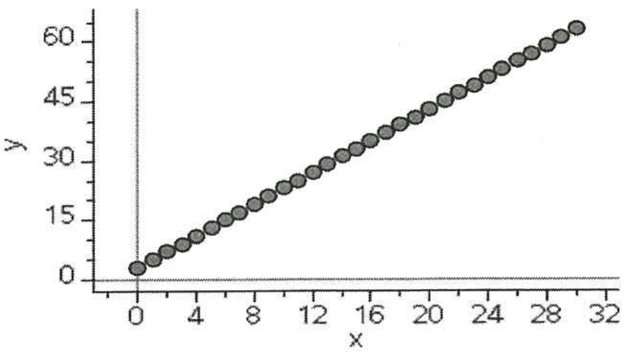
The weekly rent for the first six years will be:

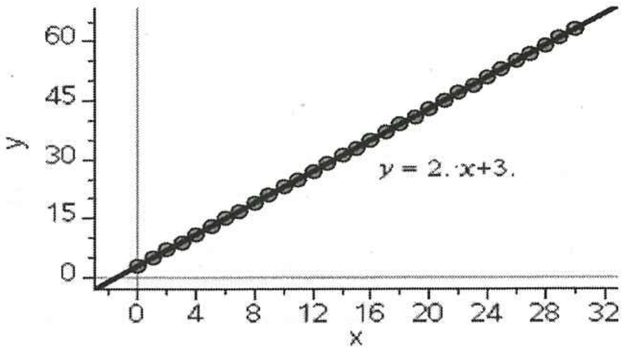
(c) Find an expression for the weekly rent r in the nth year.

CHAPTER 10 - LINEAR GRAPHS AND MODELS

10.2 Linear Functions and Graphs

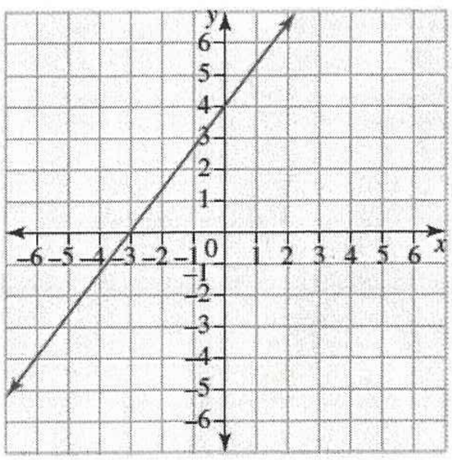
A linear function is a relationship between set of inputs and outputs of an ordered pairs x and y, {(0, 3), (1, 5), (2, 7),... (30, 63)}, joined together when graphed to form a straight line. When graphing any straight line, it is always important to represent the straight line by its rule, often refers as an *equation*. The equation for the line represents the graph below is $y = 2x + 3$.



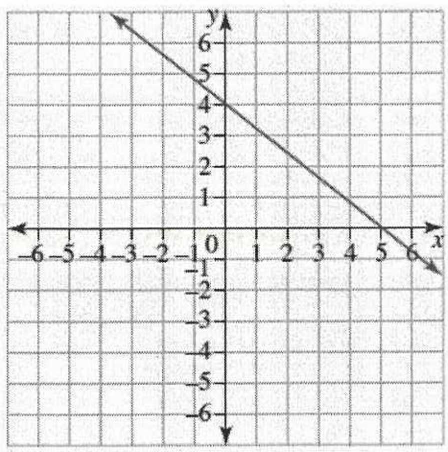


The gradient of a linear function

The gradient of a straight-line function, also known as the slope (m), determines the change in the y-value for each change in x-value.



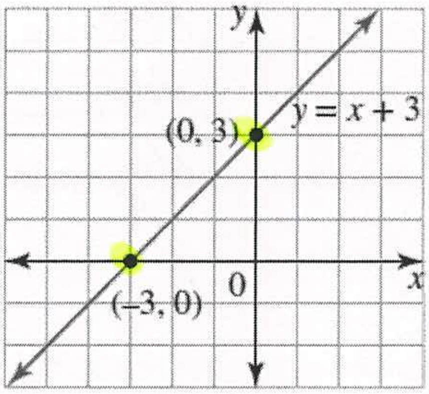
A positive gradient means that the y-value is increasing as the x-value increases



A negative gradient means that the y-value is decreasing as the x-value increases

x- and y-intercepts

- The x-intercept of a linear function is the point where the graph of the equation crosses the x-axis. This occurs when $y = 0$.
- The y-intercept of a linear function is the point where the graph of the equation crosses the y-axis. This occurs when $x = 0$.



In the graph of $y = x + 3$, we can see that the x-intercept is at $(-3, 0)$ and the y-intercept is at $(0, 3)$. These points can also be determined algebraically by putting $y = 0$ and $x = 0$ into the equation.

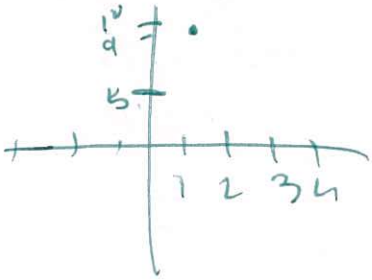
Gradient–intercept form

All linear equations can be rearranged into the gradient–intercept form,

$$y = mx + c$$

where m is the gradient and c is the y-intercept where the line cuts the y-axis

If a linear equation is in gradient-intercept form, the number in front of the x-value gives the value of the gradient of the equation. For example, in $y = 4x + 5$, the gradient is 4 and the value of the y-intercept (c) is 5.



Example 1: State the gradients and y-intercepts of the following linear equations.

Equation	Gradient	Indicating whether the gradient positive or negative	y-intercept
a) $y = 5x + 2$	5	Pos.	2
b) $y = \frac{x}{2} - 3$	$\frac{1}{2}$	Pos	-3
c) $y = -2x + 4$	-2	Neg	$+4$
d) $2y = 4x + 3$ $\frac{2y}{2} = \frac{4x}{2} + \frac{3}{2}$ $y = 2x + \frac{3}{2}$	2	Pos	Rise $\frac{3}{2}$ Run
e) $3y - 4x = 12$ $3y - 4x + 4x = 12 + 4x$ $\frac{3y}{3} = \frac{4x}{3} + \frac{12}{3}$ $y = \frac{4}{3}x + 4$	$\frac{4}{3}$	Pos	~~2~~ 4

$$\frac{3y}{3} = \frac{4x}{3} + \frac{12}{3}$$

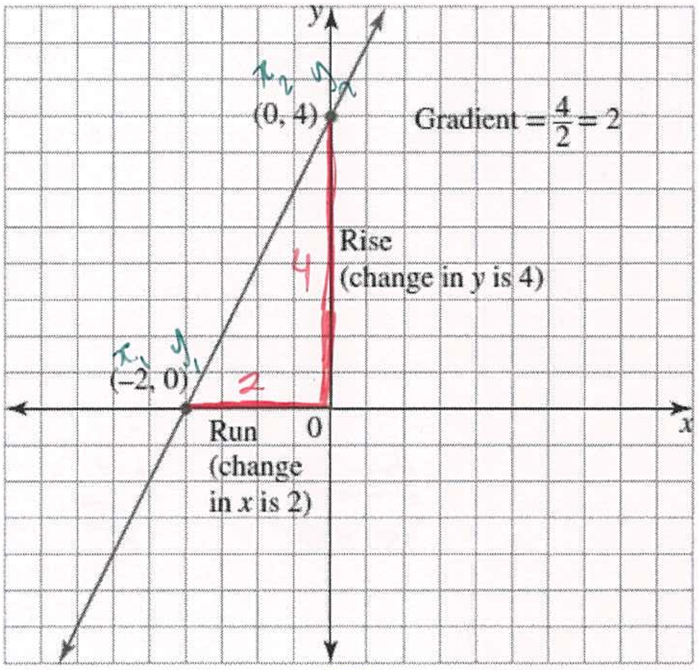
$$y = \frac{4}{3}x + 4$$

Determining the gradient from a graph

The value of the gradient can be found from a graph of a linear function. The gradient can be found by selecting two points on the line, then finding the change in the y-values and dividing by the change in the x-values.

$m = \frac{4-0}{0--2} = \frac{4}{2}$

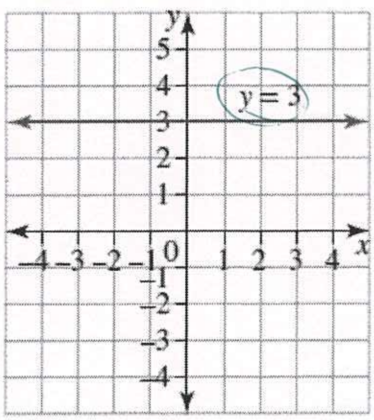
$m = 2$



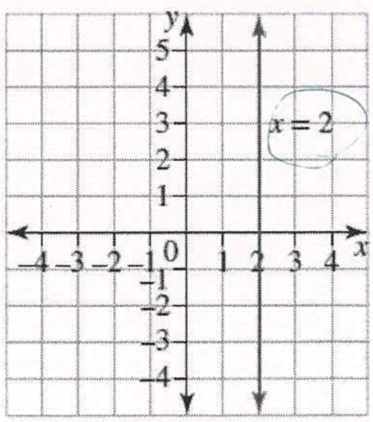
$m = \frac{4}{2}$

$m = 2$

$$\text{Gradient, } m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$



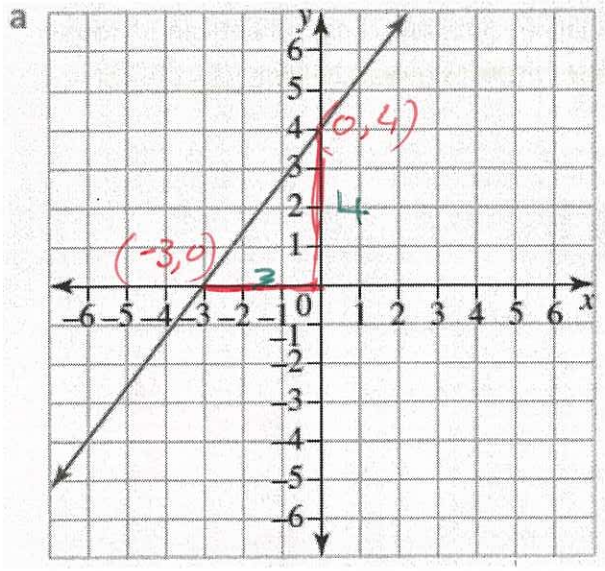
The gradient of horizontal lines is 0



The gradient of vertical lines is undefined

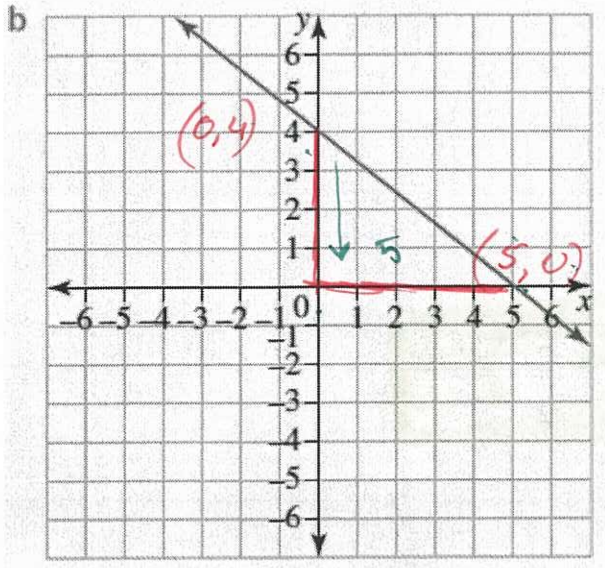
Example 2: Find the values of the gradients of the following graphs.

a



$\frac{4-0}{0--3}$ $\frac{4}{3}$

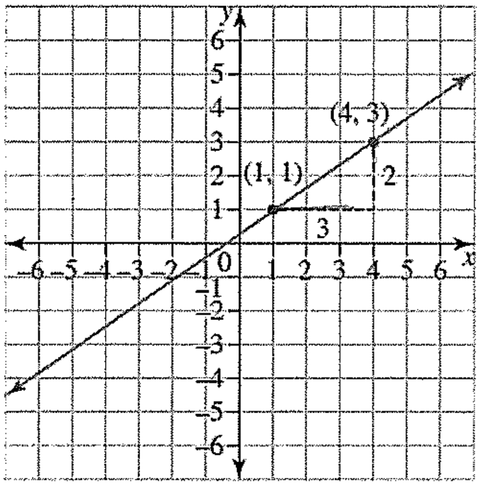
b



$\frac{0-4}{5-0} = \frac{-4}{5}$

Finding the gradient given two points

If a graph is not provided, we can still find the gradient if we are given two points that the line passes through. The same formula is used to find the gradient by finding the difference in the two y-coordinates and the difference in the two x-coordinates:



$m = \frac{2}{3}$

Example 3: Find the value of the gradients of the linear graphs that pass through the following points.

a) (4, 6) and (5, 9)

$m = \frac{9-6}{5-4} = \frac{3}{1} = 3$

b) (2, −1) and (0, 5)

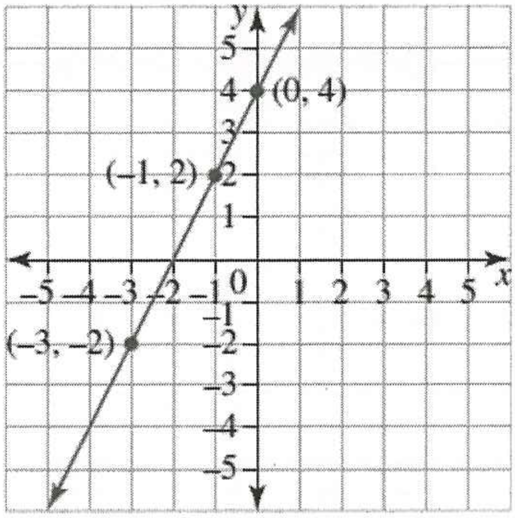
$m = \frac{5--1}{0-2} = \frac{6}{-2} = -3$

c) (0.5, 1.5) and (−0.2, 1.8)

$\frac{1.8-1.5}{-0.2-0.5} = \frac{.3}{-0.7} = -\frac{3}{7}$

Plotting linear graphs

Linear graphs can be constructed by plotting the points and then ruling a line between the points as shown in the diagram.

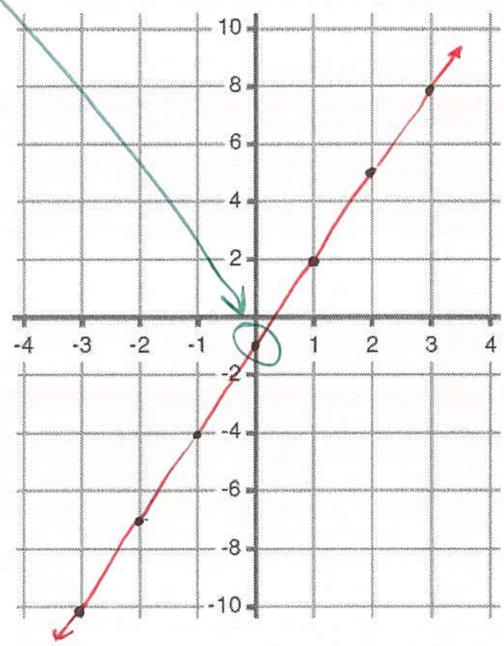


Example: Plot the graph of the linear equation $y = 3x - 1$ for values between – 3 and 3.

x	-3	-2	-1	0	1	2	3
y	-10	-7	-4	-1	2	5	8

$y = 3 \times -3 \; -1$

$y = -10$



Sketching graphs using the gradient and y-intercept method

A linear graph can be constructed by using the gradient and y-intercept. The y-intercept is marked on the y-axis, and then another point is found by using the gradient.

Example 5: Using the gradient and the y-intercept, sketch the graph of each of the following

a) A linear graph with a gradient of 3 and a y-intercept of 1

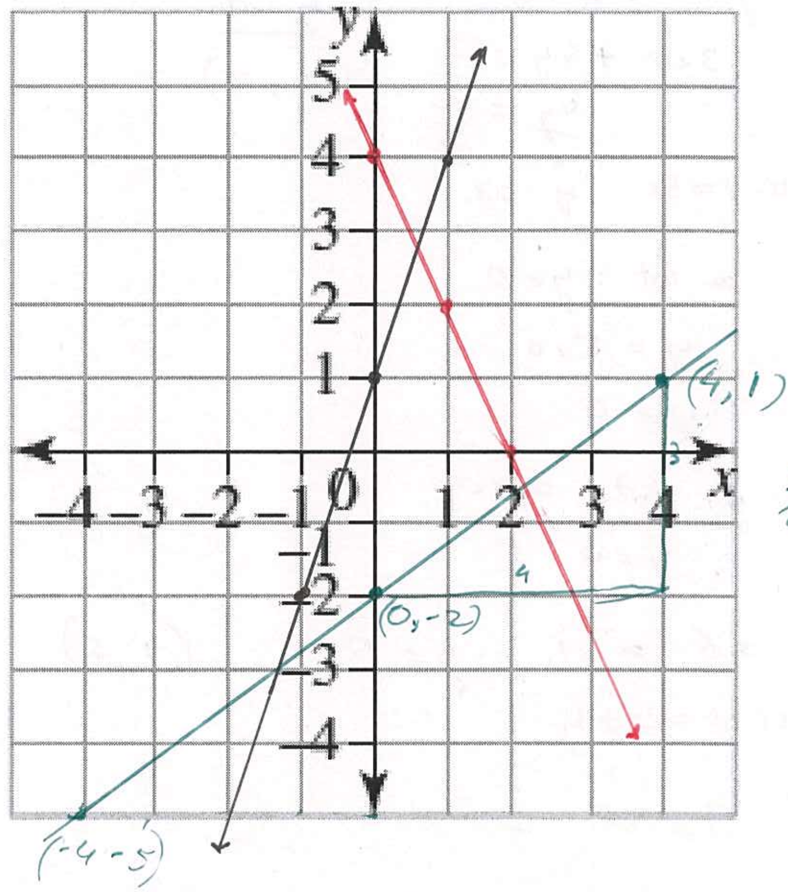
-3	-2	-1	0	1	2
-8	-5	-2	1	4	7

b) $y = -2x + 4$

c) $y = \frac{3}{4}x - 2$

$\frac{3}{4} = \frac{\text{Rise}}{\text{Run}}$

0	1
-2	



$\frac{3}{4}$ $\frac{\text{Rise}}{\text{Run}}$

Sketching graphs using the x- and y-intercepts

If the points of a linear graph where the line crosses the x- and y-axes (the x- and y-intercepts) are known, then the graph can be constructed by marking these points and ruling a line through them.

- To find the x-intercept, substitute $y = 0$ into the equation and then solve the value of x.
- To find the y-intercept, substitute $x = 0$ into the equation and then solve the value of y.

Example 6: Find the value of the x- and y- intercepts for the following linear equations, and hence sketch their graphs.

a) $3x + 4y = 12$

x-int : $y = 0$

$3x + 4 \times 0 = 12$

$\frac{3x}{3} = \frac{12}{3}$

$x = 4$

x int

$(4, 0)$

y-int : $x = 0$

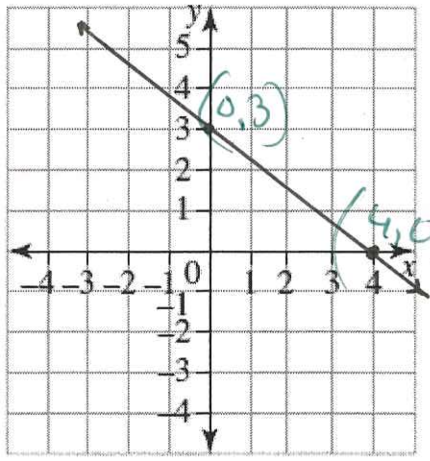
$3 \times 0 + 4y = 12$

$\frac{4y}{4} = \frac{12}{4}$

$y = 3$

y int

$(0, 3)$



b) $y = 5x$

x int : $y = 0$

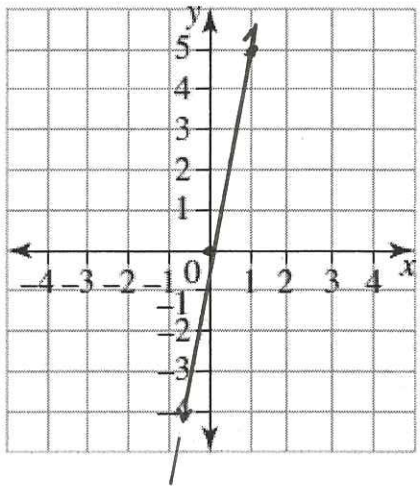
$y = 5 \times 0$

$y = 0$

y : int : $x = 0$

$y = 0$

sub $x = 1$ $\quad y = 5 \times 1$ $\quad (1, 5)$



c) $3y = 2x + 1$

$3 \times 0 = 2x + 1$

$-1 \qquad -1$

$\frac{-1}{2} = \frac{2x}{2}$

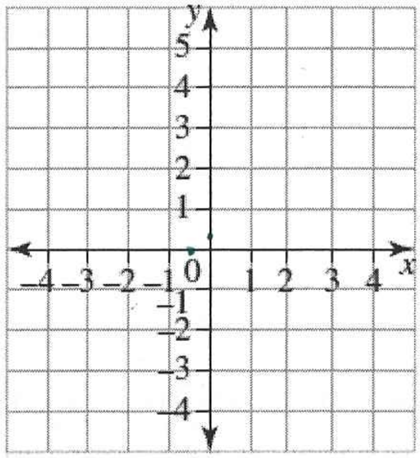
$(-\frac{1}{2}, 0)$

$-\frac{1}{2} = x$

$3y = 2 \times 0 + 1$

$\frac{3y}{3} = \frac{-1}{3}$

$(0, \frac{1}{3})$



10.3 Linear Modelling

Linear Models

Practical problems in which there is a constant change (rate of change) over time can be modelled by linear equations. The constant change, such as the rate at which water is leaking or the hourly rate charged by a tradesperson, can be represented by the gradient of the equation. Usually the y-value is the changing quantity and the x-value is time.

Example 7: Elle is an occupational therapist who charges an hourly rate of $35 on top of an initial charge of $50. Construct a linear equation to represent Elle's charge, C, for a period of t hours.

$C = 35t + 50$

Solving practical problems

Once an equation is found to represent the practical problem, solutions to the problem can be found by sketching the graph and reading off important information such as the value of the x- and y-intercepts and the gradient. Knowing the equation can also help to find other values related to the problem.

Interpreting the parameters of linear models

When we have determined important values in practical problems, such as the value of the intercepts and gradient, it is important to be able to relate these back to the problem and to interpret their meaning.

Example 8: A bike tyre has 500 cm3 of air in it before being punctured by a nail. After the puncture, the air in the tyre is leaking at a rate of 5 cm^3/minute.

a) Construct an equation to represent the amount of air, A, in the tyre t minutes after the puncture occurred.

$A = 500 - 5t$

or

$A = -5t + 500$

b) Interpret what the value of the gradient in the equation means.

Every minute the air leaks $5 cm^3$

c) Determine the amount of air in the tyre after 12 minutes.

$A = 500 - \cancel{t}\ 5 \times 12$

$= 500 - 60$

$= 440\ cm^3$

There are $440\ cm^3$ of air left after 12 minutes

d) By solving your equation from part a, determine how long, in minutes, it will take before the tyre is completely flat (i.e. there is no air left).

$0 = -5t + 500$

$-500 \qquad -500$

$\frac{-500}{-5} = \frac{-5t}{-5}$

$100 = t$

After 100 min the tyre will be flat.

10.4 Linear equations and predictions

Finding the equation of straight lines

GIVEN THE GRADIENT AND Y-INTERCEPT

When we are given the gradient and y-intercept of a straight line, we can enter these values into the equation $y = mx + c$ to determine the equation of the straight line.

GIVEN THE GRADIENT AND ONE POINT

When we are given the gradient and one point of a straight line, we need to establish the value of the y-intercept to find the equation of the straight line. This can be done by substituting the coordinates of the given point into the equation $y = mx + c$ and then solving for c.

GIVEN TWO POINTS

When we are given two points of a straight line, we can find the value of the gradient of a straight line between these points as discussed in Section 10.2 (by using $m = \frac{y_2 - y_1}{x_2 - x_1}$). Once the gradient has been found, we can find the y-intercept by substituting one of the points into the equation $y = mx + c$ and then solving for c.

Example 10: Find the equations of the following straight lines.

a) A straight line with a gradient of 2 passing through the point (3, 7)

$m = 2$

$y = 2x + c$ $\quad \begin{pmatrix} 3 & 7 \\ x & y \end{pmatrix}$

so

$7 = 2 \times 3 + c$

$7 = 6 + c$

$1 = c$

b) A straight line passing through the points (1, 6) and (3, 0)

$\frac{0-6}{3-1} = \frac{-6}{2} = -3$ $\quad y = -3x + c$

$6 = -3 \times 1 + c$

$6 = -3 + c$

$9 = c$

c) A straight line passing through the points (2, 5) and (5, 5)

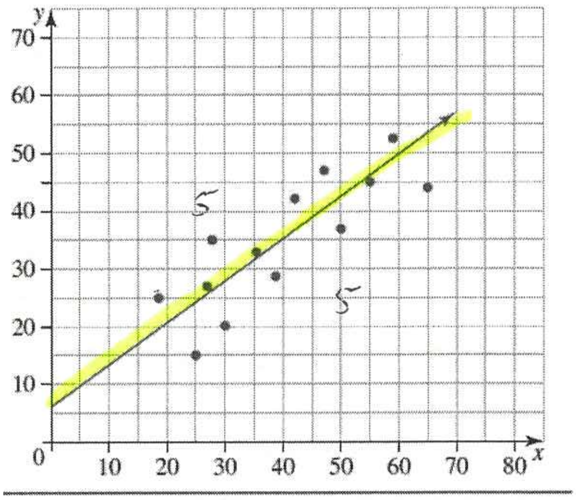
$\frac{5-5}{5-2} = \frac{0}{3} = 0$ $\quad y = c$

$5 = c$ $\quad y = 5$

Lines of best fit

Sometimes the data for a practical problem may not be in the form of a perfect linear relationship, but the data can still be modelled by an approximate linear relationship.

When we are given a scatterplot representing data that appears to be approximately linear, we can draw a *line of best fit* by eye so that approximately half of the data points are on either side of the line of best fit.



After drawing a line of best fit, the equation of the line can be determined by picking two points on the line and determining the equation, as demonstrated in the previous section. This method is not as reliable as there are so many variation of how the line of best fit is drawn. A more reliable method of determining the line of best fit is to use mathematical procedure, known as the *least square method*. For simplification we will use technology, CAS calculator, to help us determining the equation for the line of best fit.

Creating a line of best fit using CAS calculator

Example 11: The table below represent the relationship between the test scores in Mathematics and Physics for ten Year 11 students. Draw a scatterplot and find the equation for the line of best fit represents for this set of data.

Test score in Mathematics	65	43	72	77	50	37	68	89	61	48
Test score in Physics	58	46	78	83	35	51	61	80	55	62

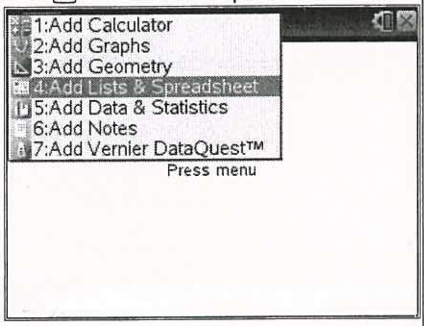
Prior to drawing the scatterplot, we need to decide which variable would go on the x-axis and which would go on the y-axis. Generally, the response (dependent) variable would go on the y-axis and the explanatory (independent) variable would go on the x-axis. In this situation, one would argue that to be able to well in the Physics Test, one must perform well in the Mathematics Test well as Physics concepts involves a lots of mathematics concepts. Hence:

- Explanatory/Independent variable (x-axis) is Test Score in Maths
- Response/Dependent variable (y-axis) is Test Score in Physics

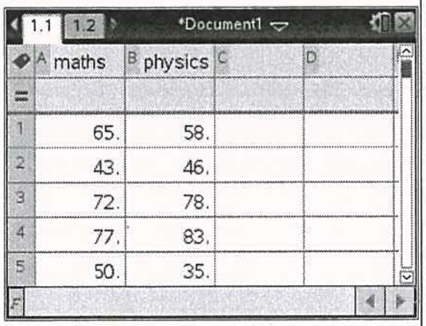
Example 11 on CAS calculator

Start with a blank calculator page. Press

- [⌂on] Home
- [1] New document
- [1] Add Lists & Spreadsheet

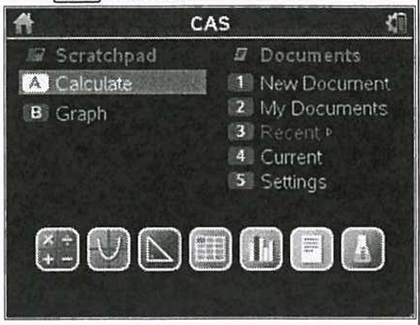


- Label column A as "maths"
- Label column B as "physics"
- Enter the scores in each column respectively.
-



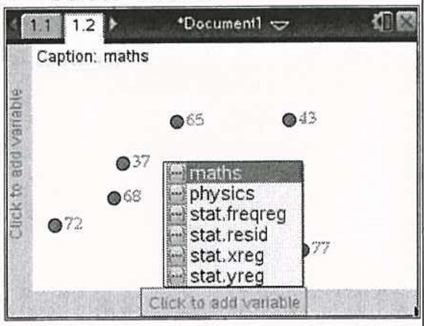
Select the

- [⌂on] Home

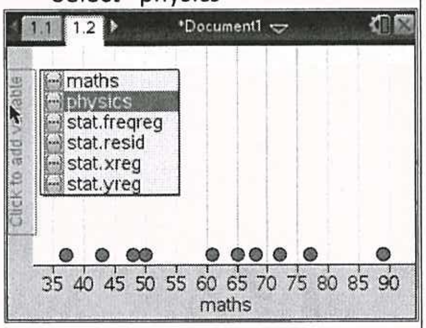


- Add Data & Statistics icon

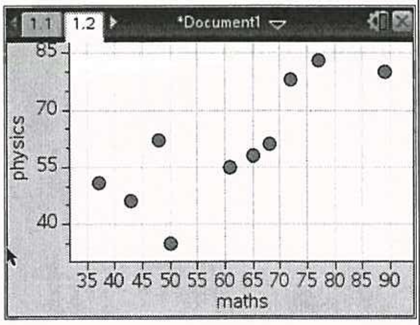
- Along the x-axis "Click to add variable"
- Select "maths"



- Along the y-axis "Click to add variable"
- Select "physics"

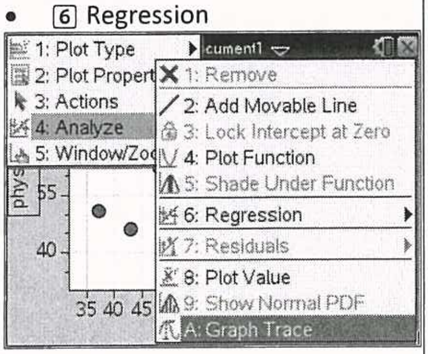


The scatterplot will appear as followed:



Select:

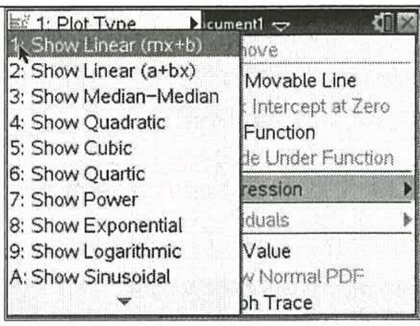
- [menu] Menu
- [4] Analyse
- [6] Regression

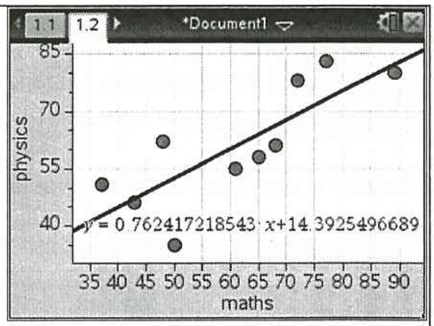


Select:

- [1] Show Linear (mx+b)

The equation of the line of best fit through scatterplot will appear as followed:





The equation that we see from the CAS calculator is:

$y = 0.76x + 14.39$

However this equation does not replicate the situation we look at. We need to write the equation that actually represent the situation by using the correct variables.

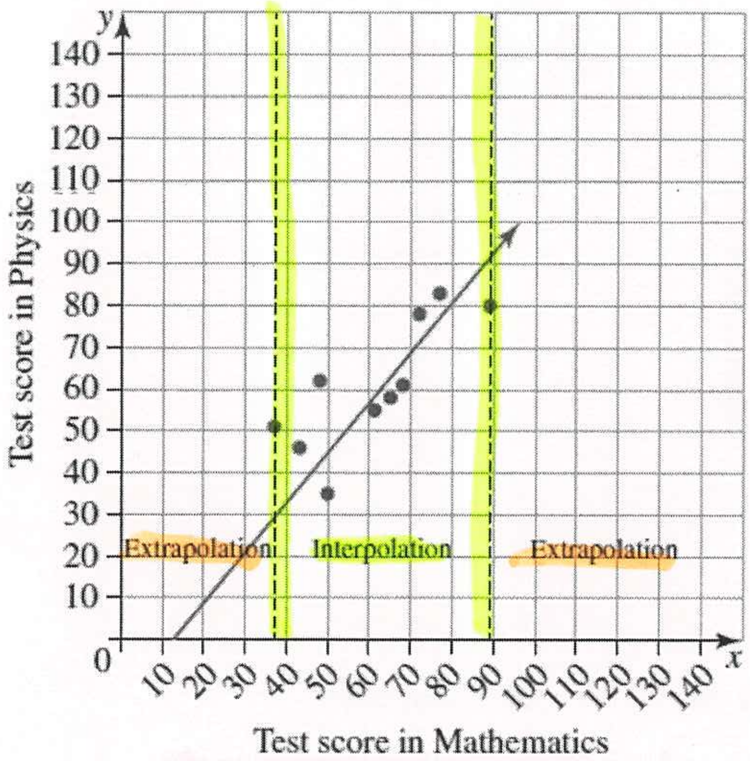
Making Predictions

INTERPOLATION

When we use ***interpolation***, we are making a prediction from a line of best fit that appears within the ranges of the original data set.

EXTRAPOLATION

When we use ***extrapolation***, we are making a prediction from a line of best fit that appears outside the ranges of the original data set.

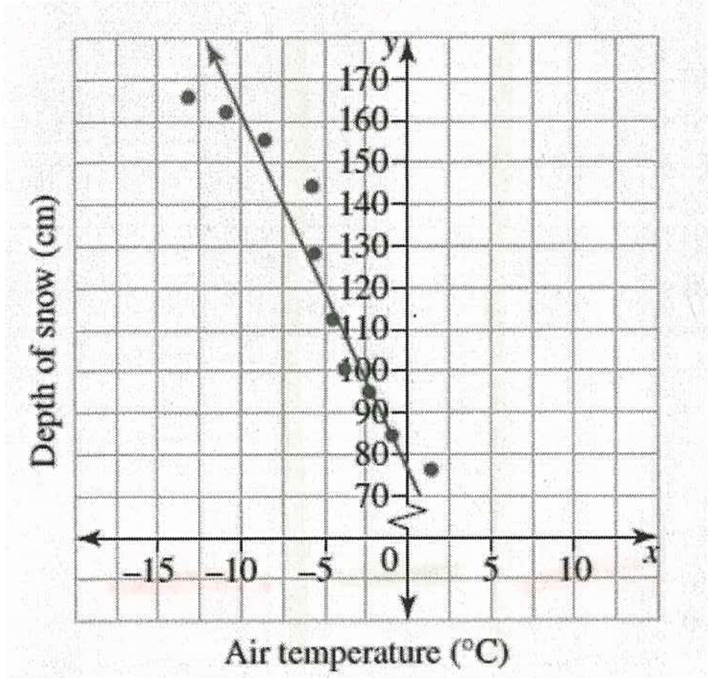


RELIABILITY OF PREDICTIONS

- The more pieces of data there are in a set, the better the line of best fit you will be able to draw. More data points allow more reliable predictions.
- Interpolation is a far more reliable method of making predictions than extrapolation.

Example 12: The following data represent the air temperature (°C) and depth of snow (cm) at a popular ski resort.

Air temp (°C)	−4.5	−2.3	−8.9	−11.0	−13.3	−6.2	−0.4	1.5	−3.7	−5.4
Depth of snow (cm)	111.3	95.8	1555.6	162.3	166.0	144.7	84.0	77.2	100.5	129.3



The line of best fit for this data set has been calculated as

$$\textit{Depth of snow} = -7.2 \times \textit{Air temp} + 84$$

a) Use the line of best fit to estimate the depth of snow if the air temperature is $-6.5°$ C.

Depth of snow $= -7.2 \times -6.5 + 84$

$= 130.8$

When temp is -6.5 the snow depth will be 130.8 cm

b) Use the line of best fit to estimate the depth of snow if the air temperature is $25.2°$ C.

$x = 25.2$

$y = -7.2 \times 25.2 + 84$

$= -97.4$

When temp is 25.2 the snow depth will be -97.4 cm

c) Comment on the reliability of your estimations in parts a and b.

The estimation for a seems reliable as it is interpolated and sits accurately within the data. Estimate b is extrapolated which means it is not reliable