6.3 Compound Interest Tables

For investments, when interest is added to the initial amount (principal) invested at the end of an interest-bearing period, and then both the principal and interest earn further interest during the next period, which in turn is added to the balance. This process continues for the life of the investment. The interest is said to be compounded.

Both the balance of the account and interest increase at regular intervals.

Example

Consider $1000 invested for 4 years at an interest rate of 12% p.a. with interested compounded annually. What will be the final balance of the account?

Time period ($n+1$)	V_n($)	Interest ($)	V_{n+1} ($)
1	V_0 = 1000.00	12% of 1000 = 120.00	1000+120 = 1120.00 ← V_1
2	V_1 = 1120.00	12% of 1120 = 134.40	1120+134.40=1254.40 ← V_2
3	V_2 = 1254.40	12% of 1254.40 = 150.53	1254.40+150.53 = 1404.93 ← V_3
4	V_3 = 1404.93	12% of 1404.93 = 168.59	1404.93+168.59 = 1573.52 ← V_4
5	V_4 = 1573.52	12% of 1573.52 = 188.82	1573.52+188.82 = 1762.34 ← 5

So the balance after 5 years is $1762.34.

In the above example the principle is increased by 12% per year. That is at the end of year balance is 112% or 1.12 of the start of year balance. → So, we could also do it like this

The 1.12 is called "the compounding factor."

Time period	Balance($)	
1	1120 = 1000 × 1.12 = 1000 · 1.12	= 1000 $(1.12)^1$
2	1254.40 = 1120 × 1.12 = 1000 × 1.12 × 1.12	= 1000 $(1.12)^2$
3	1404.93 = 1254.40 × 1.12 = 1000 × 1.12 × 1.12 × 1.12	= 1000 $(1.12)^3$
4	1573.52 = 1404.93 × 1.12 = 1000 × 1.12 × 1.12 × 1.12 × 1.12	= 1000 $(1.12)^4$
5	1762.34 = 1573.52 × 1.12 = 1000 × 1.12 × 1.12 × 1.12 × 1.12 × 1.12	= 1000 $(1.12)^5$

If this investment continued for n years, the final balance should be:

$$V_n = 1000\,(1.12)^n = 1000\,(1+0.12)^n = 1000\left(1+\frac{12}{100}\right)^n$$

Worked Example 6

Laura invested $2500 ($V_0$) for 5 years at an interest rate of 8% p.a. with interest compounding annually. Complete the table by calculating the values A, B, C, D, E and F.

Time period ($n+1$)	V_n ($)	Interest ($)	V_{n+1} ($)
1	2500	8 % of 2500 = 200	2700
2	2700	8% of 2700 = 216	2700 + 216 = 2916
3	2916	8% of 2916 = 233.28	3149.28
4	3149.28	8% of 3149.28 = 251.94	3149.28 + 251.94 = 3401.22
5	3401.22	8% of 3401.22 = 272.10	3673.32

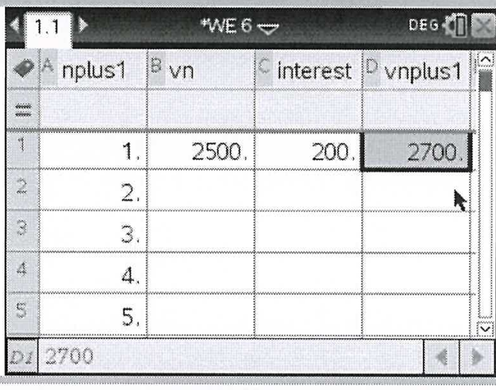
Worked Example 6 on CAS calculator

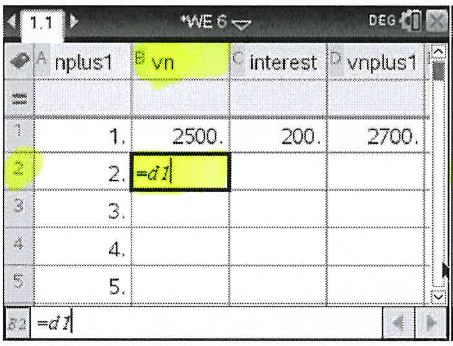
Enter the labels "n+1", "V_n", "Interest", "V_{n+1}"

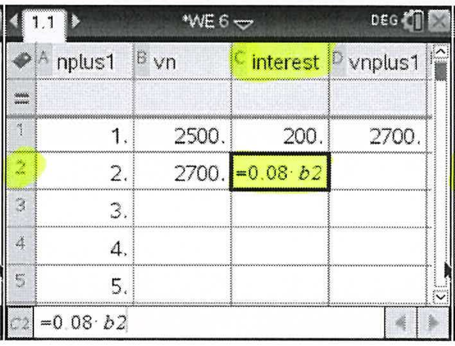
Note: You can't use + on the CAS so spell it out

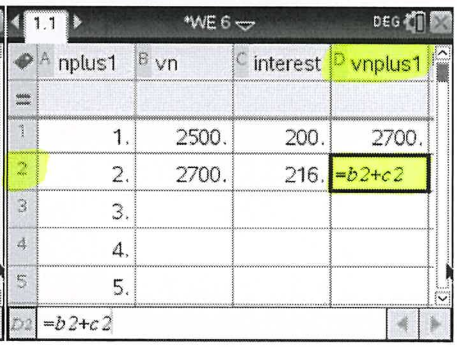
Next enter 1 to 5 in column A, and the starting values for V_n=2500, Interest=200 and V_{n+1}=2700 in cells b1, c1 and d1 respectively.

Then enter formulas shown below into cells b2, c2 and d2

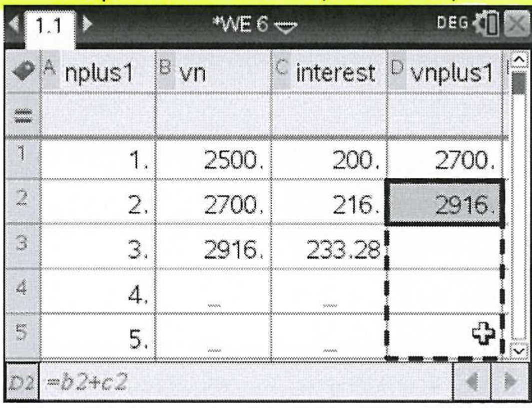


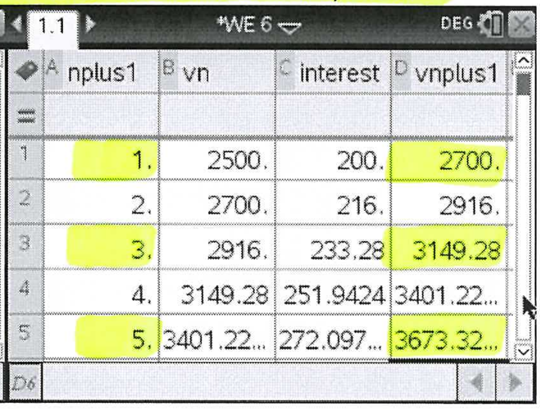






Now fill down the equations of cells b2, c2 and d2, downward for each of columns b, c and d.





The last screen picture shows the completed table.

6.4 Compound interest formula

From the previous section (6.3) we can see that we could write the value of the investment in terms of its previous value and hence, express it as the recurrence relation:

$$V_{n+1} = V_n R$$

where V_{n+1} is the amount of the investment 1 time period after V_n, R is the growth or compounding factor $\left(= 1 + \frac{r}{100}\right)$ and r is interest rate per period.

This pattern can be written in terms of the initial investment. This is the compound interest formula.

$V_n = V_0 R^n$ where
V_n = final or total amount ($)
V_0 = principal ($)
R = growth or compounding factor $\left(= 1 + \frac{r}{100}\right)$
r = interest rate per period
n = number of interest-bearing periods

This formula gives the total amount in an account, not just the interest earned.

To find the total interest compound, *I*:

$I = V_n - V_0$ where
V_n = final or total amount ($)
V_0 = principal ($)

Worked Example 7

$5000 is invested for 4 years at 6.5% p.a., interest compound annually.

a) Generate the compound interest formula for this investment.

$V_0 = \$5000$
$n = 4$ years
$r = 6.5\%$

$V_n = V_0 \times R^n, \quad R = \left(1 + \frac{r}{100}\right)$

$V_n = 5000\left(1 + \frac{6.5}{100}\right)^n$

simplify, $V_n = 5000(1.065)^n$

b) Find the amount in the balance after 4 years and the interest earned over this period.

$n = 4$

$V_4 = 5000(1.065)^4$
$= \$6432.33$ ← balance after 4 years

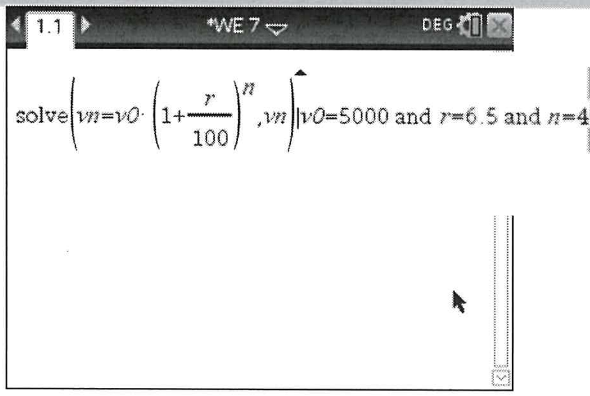
$I = V_4 - V_0 = \$6432.33 - \$5000$
$= \$1432.33$ ← interest earned.

Worked Example 7 on CAS calculator

On a calculator page
Using the Solve function
Enter the compound interest formula

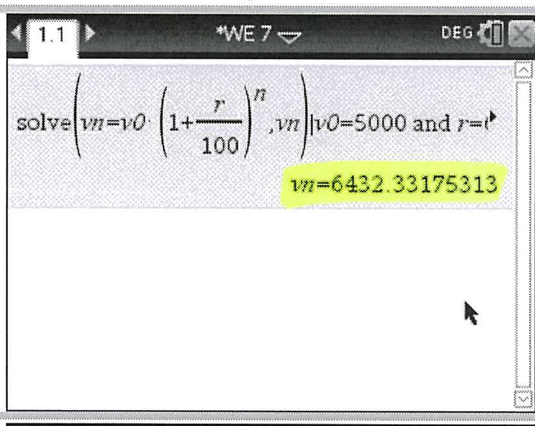
$$V_n = V_o\left(1+\frac{r}{100}\right)^n$$

and set the values of
V_0=$5000, r=6.5 and n=4
using "|*"



Top Tip: You could save this document on your CAS and just change the values

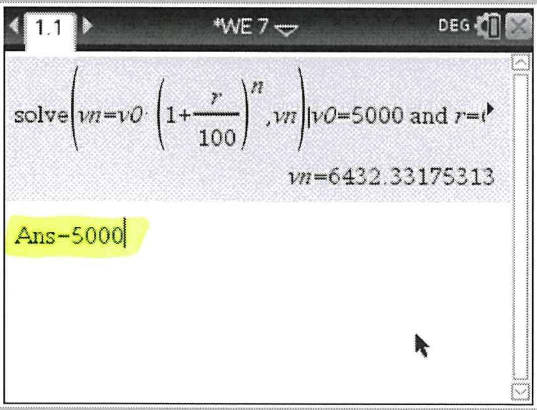
Press [enter] to get the value of V_n



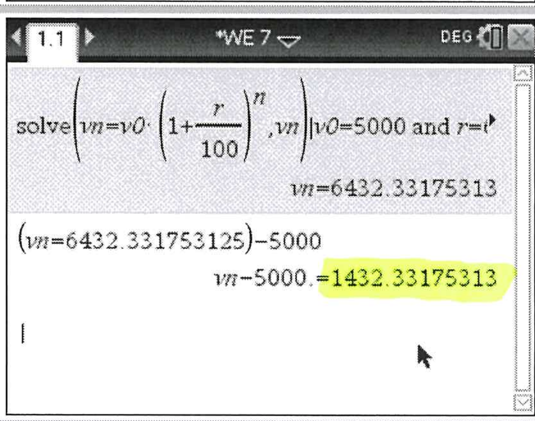
To find the interest earned, subtract the principal from the balance. (V_n-V_0)

On the CAS enter
[-][5][0][0][0]

(note: the CAS will insert ANS before the minus)



Press [enter] to get the answer



*| tells the CAS the values of variables, think of line entry as:
"**solve this**" (equation with variables) "**when**" the variables are…

solve(equation, variable) | x = __ and y = __

Non-annual compounding

Many accounts can be compounded quarterly (every three months), weekly or daily. In these cases *n* and *r* are determined as follows:

Number of interest periods, n = number of years × number of interest periods per year

$$\text{Interest rate per period, } r = \frac{\text{nominal interest rate per annum}}{\text{number of interest periods per year}}$$

Nominal interest rate per annum is the annual interest rate advertised by a financial institution.

Worked Example 8

If $3200 is invested for 5 years at 6% p.a., interest compounded quarterly:

a) Find the number of interest bearing periods, *n*

4 quarters per year, for 5 years

$\Rightarrow$ $n = 4\text{ (quarters)} \times 5\text{ (years)} = 20$

b) find the interest rate per period, *r*

$r = \frac{6\%\text{ p.a.}}{4}$ ← per year

$= 1.5\%$ per quarter

c) find the balance of the account after 5 years

$V_n = \$3200$

$n = 20$

$r = 1.5\%$

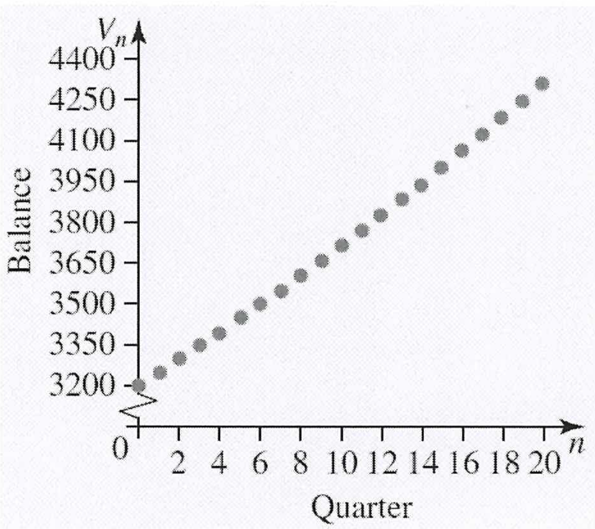
$V_{20} = \$3200\left(1 + \frac{1.5}{100}\right)^{20}$

$= \$3200(1.015)^{20}$

$= \$4309.94$

Balance of account after 5 years is $4309.94

d) graphically represent the balance at the end of each quarter for 5 years. Describe the shape of the graph.



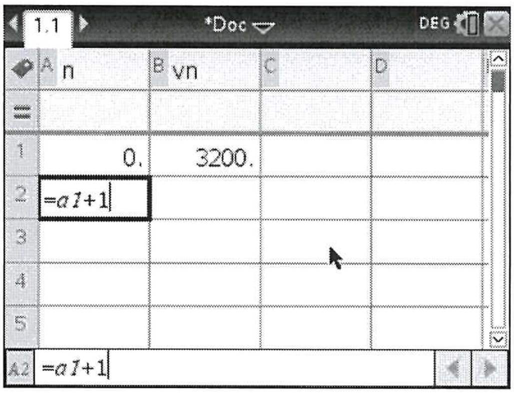
The graph is exponential as the interest is added at the end of each quarter and the following interest is calculated on the new balance.

The graph is exponential as the interest is added at the end of each quarter and the following interest is calculated on the new balance.

Worked Example 8(c) and 8(d) on CAS calculator

Top Tip: Because we want to create a graph in 8(d) we will do this on a "list & spreadsheet" page

- On a "list & spreadsheet" page
- Label column A "n" and column B "Vn"
- In cell *a1* enter "0" and in *b1* enter the V_0 value of $3200
- In cell *a2* enter the formula =*a1*+1, and then fill down ([menu][3][3]) to cell *a21* (from 8(a) n=20)

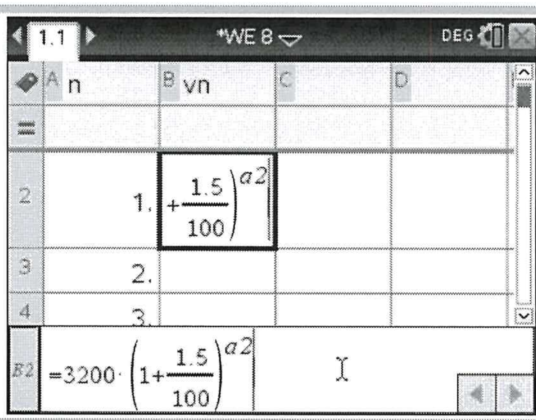


- In cell *b2* enter the formula

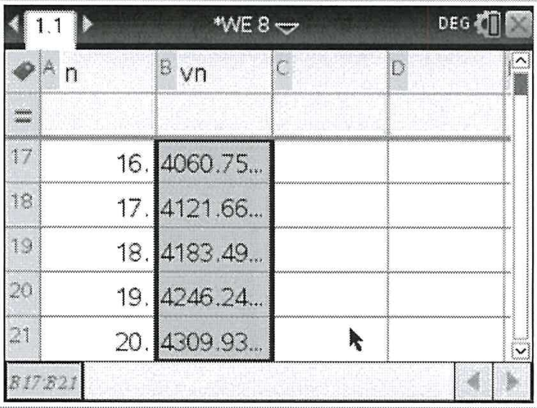
$$= 3200\left(1 + \frac{1.5}{100}\right)^{a2}$$

Note: r=1.5 is from part (b) $r = \frac{6}{4}$

- Press [enter] to get the value of V_1

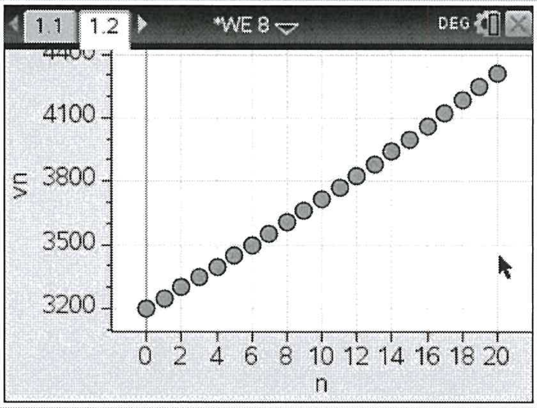


- Then fill down ([menu][3][3])



- Add a "data & statistics" page ([ctrl][doc▾])

Label the x-axis "n"(Quarters) and the y-axis "Vn" (Balance)



Worked Example 9

Find the principal that will grow to $4000 in 6 years, if interest is added quarterly at 6.5% p.a.

In this case we want to have a certain amount of money ($4000) in six years. How much do we invested?

$n = 6(\text{years}) \times 4(\text{quarters}) = 24$, $r = \frac{6.5}{4}$ p.a. $= 1.625$.

and $V_{24} = \$4000$ (amount at end)

$$V_{24} = V_0\left(1 + \frac{r}{100}\right)^n$$

$$\$4000 = V_0\left(1 + \frac{1.625}{100}\right)^{24}$$

$$\Rightarrow V_0 = \frac{4000}{(1.01625)^{24}}$$

$$V_0 = \$2716.73$$

← Principal would need to be invested.

Worked Example 9 on CAS calculator

On a calculator page

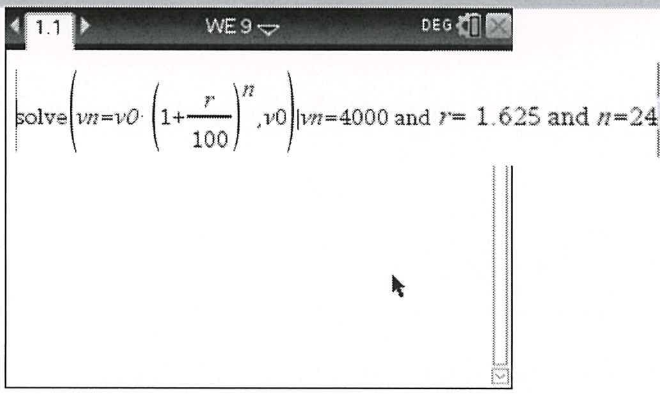
Using the Solve function

Enter the compound interest formula

$$V_n = V_o\left(1 + \frac{r}{100}\right)^n$$

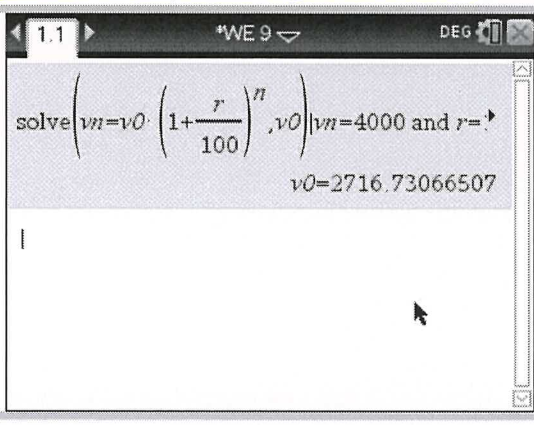
and set the values of

V_0=$4000, r=1.625 and n=24 using "|"



Top Tip: You could save this document on your CAS and just change the values

Press enter to get the value of V_0



Modelling Geometric Growth and Decay

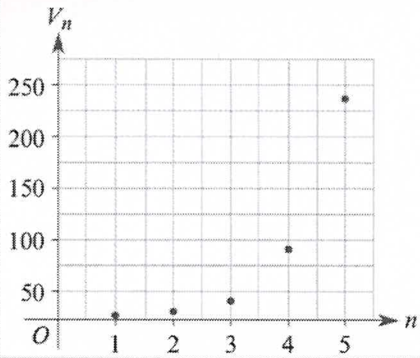
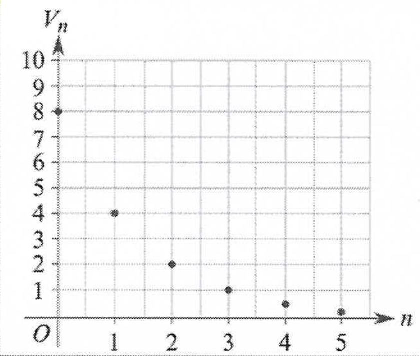
Compound interest is a geometric growth or decay. That is, it is a non-linear growth or decay. This is due to the fact that the *rule* governing compound interest is an increase (or decrease) by the same rate/Interest (as a percentage) at regular intervals.

Consider the following recurrence relations:

$$V_0 = 1, \quad V_{n+1} = 3V_n$$

$$V_0 = 8, \quad V_{n+1} = 0.5V_n$$

both have rules that generate a geometric pattern shown below

Recurrence relation	*Rule*	*Sequence*	*Graph*
$V_0 = 1,$ $V_{n+1} = 3V_n$	'multiply by 3'	$1, 3, 9, \ldots$	
$V_0 = 8,$ $V_{n+1} = 0.5V_n$	'multiply by 0.5'	$8, 4, 2, \ldots$	

The first generates a sequence whose terms grow geometrically and the second one decays geometrically.

In general, the rule:

$$V_{n+1} = RV_n$$

- is geometric growth if $R>1$
- is geometric decay if $R<1$

where R is the growth or compounding factor $\left(= 1 + \frac{r}{100}\right)$ and r is the interest rate per period.