5.3 First-order linear recurrence relations

First-order linear recurrence relations with a common difference

The **common difference**, d, is the value between consecutive terms in the sequence:

Look at the sequence 3, 7, 11, 15, 19, ...

$$d = u_2 - u_1 = u_3 - u_2 = u_4 - u_3 = \ldots$$

$$d = 7 - 3 = 11 - 7 = 15 - 11 = +4$$

The common difference is +4.

This sequence may be defined by the first-order linear recurrence relation:

$$u_{n+1} - u_n = 4 \qquad u_0 = 3$$

Rewriting this

$$u_{n+1} = u_n + 4 \qquad u_0 = 3$$

A sequence with a common difference of d may be defined by a first-order linear recurrence relation of the form:

$$u_{n+1} = u_n + d \qquad (\text{or } u_{n+1} - u_n = d)$$

where d is the common difference and for

$d > 0$ it is an increasing sequence

$d < 0$ it is a decreasing sequence.

Worked Example 4

Express each of the following sequences as first-order recurrence relations.

a) 7, 12, 17, 22, 27, ...

$u_0 = 7$

$$\left.\begin{aligned} d &= 12 - 7 = 5 \\ d &= 17 - 12 = 5 \\ d &= 22 - 17 = 5 \\ d &= 27 - 22 = 5 \end{aligned}\right\} \text{ so } d = 5.$$

So, $u_0 = 7$, $u_{n+1} = u_n + 5$

b) 9, 3, -3, -9, -15, ...

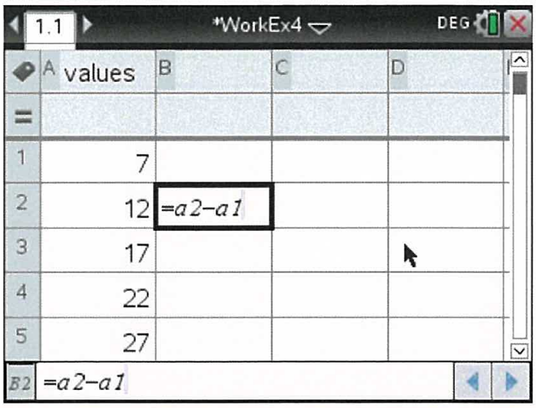
$u_0 = 9$

$$\left.\begin{aligned} d &= 3 - 9 = -6 \\ d &= -3 - 3 = -6 \\ d &= -9 - 3 = -6 \\ d &= -15 - 9 = -6 \end{aligned}\right\} d = -6.$$

So $u_0 = 9$ and $u_{n+1} = u_n - 6$.

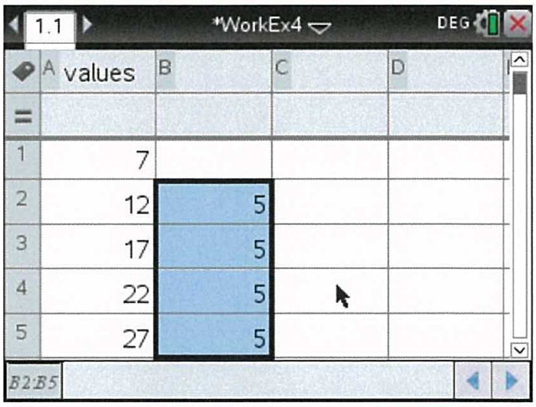
Worked Example 4: Using CAS Calculator

To check if there is a common difference



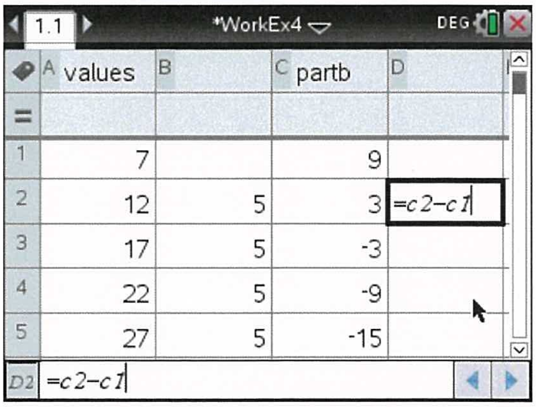
Use a list and spreadsheet page

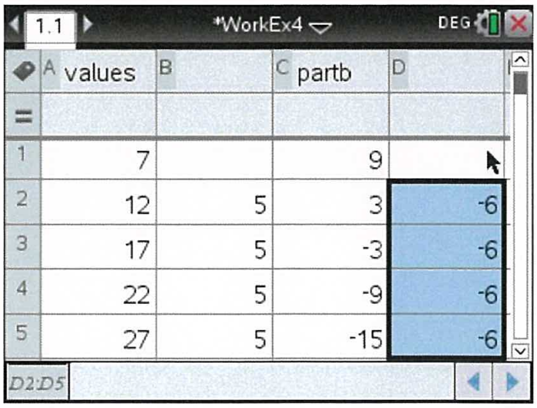
- Enter the values in the first column



- Enter the equation in column B
- Check all values are the same

Repeat for Part b





First-order linear recurrence relations with a common ratio

Not all sequences have a **common difference** (increasing/decreasing by adding/subtracting the same difference to find the next term).

The sequence may increase/decrease by multiplying the terms by a **common ratio**.

Look at the geometric sequence 1, 3, 9, 27, 81, …

The common ratio can be found by dividing the current term by the previous term. So generally:

$$R = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \cdots$$

And in this example:

$$R = \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \cdots$$

Here the common ratio is 3.

The sequence can be defined by the first-order linear recurrence relation:

$u_{n+1} = 3u_n$ where: $u_0 = 1$

> **A sequence with a common ratio of R may be defined by a first-order linear recurrence relation of the form:**
>
> $$u_{n+1} = Ru_n$$
>
> **where R is the common ratio**
>
> **$R > 1$ is an increasing sequence**
>
> **$0 < R < 1$ is a decreasing sequence**
>
> **$R < 0$ is a sequence alternating between positive and negative values.**

Worked Example 6

Express each of the following sequences as first-order recurrence relations.

a) 1, 5, 25, 125, 625, ….

$u_0 = 1$ $R = \frac{5}{1} = \frac{25}{5} = \frac{125}{25} = \frac{625}{125} = 5$

so $R = 5$

$u_0 = 1$ $u_{n+1} = 5u_n$

starting point rule.

b) 3, -6, 12, -24, 48, …

$u_0 = 3$ $R = \frac{-6}{3} = \frac{12}{-6} = \frac{-24}{12} = \frac{48}{-24}$

$= -2$

so $u_0 = 3$ $u_{n+1} = -2u_n$

Example 7

State the value of R and write down the first 5 terms of the following sequences from the first-order recurrence relations.

a) $u_0=2$, $u_{n+1} = 7u_n$ $n = 0, 1, 2, 3, 4, \ldots$

$u_0 = 2$

$u_{n+1} = 7u_n$

when $n=0$

$u_{0+1} = 7 \times u_0$

$u_1 = 7 \times 2 = 14$

when $n=1$ $u_{1+1} = 7u_1$

$u_2 = 7 \times 14$

$u_2 = 98$

when $n=2$, $u_3 = 7 \times u_2$

$= 7 \times 98$

$u_3 = 689$

$u_4 = 7 \times u_3$

$= 7 \times 689$

$u_4 = 4802$

so, 2, 14, 98, 689, 4802

b) $u_0=1$, $u_{n+1} = -\frac{1}{2} u_n$ $n = 0, 1, 2, 3, 4, \ldots$

$u_0 = 1$

$u_1 = -\frac{1}{2} \times u_0$

$u_1 = -\frac{1}{2}$

$u_2 = -\frac{1}{2} \times u_1$

$u_2 = -\frac{1}{2} \times -\frac{1}{2}$

$= \frac{1}{4}$

$u_3 = -\frac{1}{2} \times u_2$

$u_3 = -\frac{1}{2} \times \frac{1}{4}$

$= -\frac{1}{8}$

$u_4 = -\frac{1}{2} \times u_3$

$= -\frac{1}{2} \times -\frac{1}{8}$

$= \frac{1}{16}$

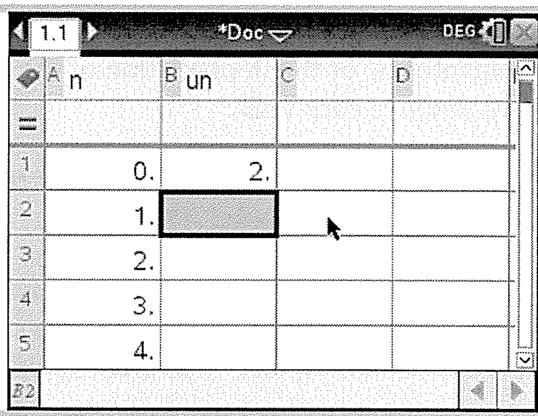
so the sequence is

$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$

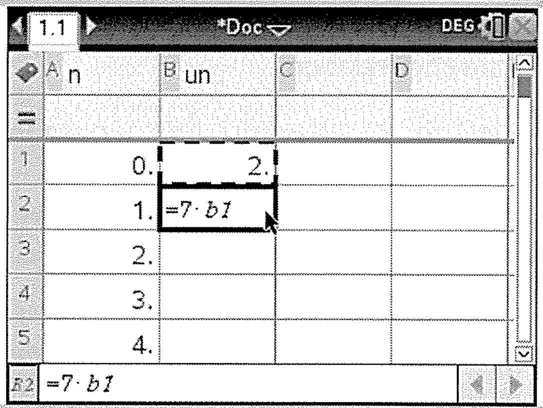
Example 7 Using CAS Calculator

Part a: $u_0=2$, $u_{n+1} = 7u_n$ $n = 0, 1, 2, 3, 4, \ldots$

- Label column A "n"
- Enter the n values of 0 to 4 into column A
- Label column B as u_n
- Enter the value of u_0 (=2) into cell b1

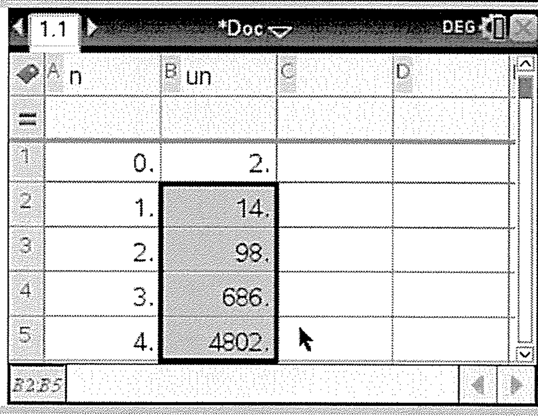


- Label Column B "value"
- Enter the equation for u_{n+1} into cell b2 after an = sign
- Press [enter]



Now fill down this equation to the cells below.
Press

- Menu [menu]
- data [3]
- fill [3]



Modelling linear growth and decay

Linear growth and decay is commonly found around the world. They occur when a quantity increases or decreases by the same amount at regular intervals. Everyday examples include the paying of simple interest or the depreciation of the value of a new car by a constant amount each year.

An example of linear growth is the **investment of money**, such as putting it in a savings account where the sum increases over time.

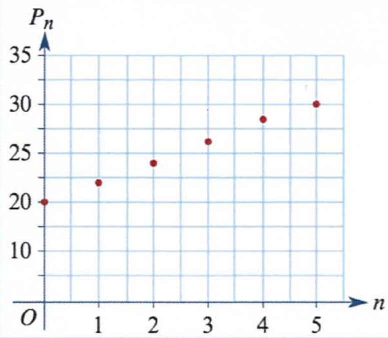
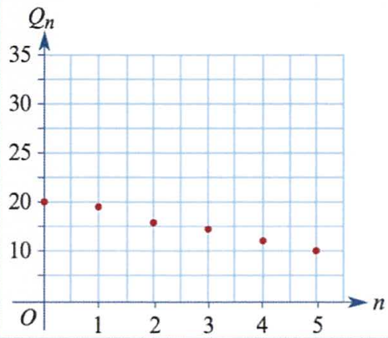
An example of linear decay is the **money owned** to repay a loan, the sum of money owned will decrease over time. (an example of which is the "Holiday ghost" – Nimble loan ad)

A recurrence model for linear growth and decay

The recurrence relations

$$P_o = 20,\ P_{n+1} = P_n + 2 \qquad Q_o = 20,\ Q_{n+1} = Q_n - 2$$

both have rules that generate sequences with linear patterns, as can be seen from the table below. The first generates a sequence whose successive terms have a linear pattern of growth, and the second a linear pattern of decay.

Recurrence relation	*Rule*	*Sequence*	*Graph*
$P_o = 20, P_{n+1} = P_n + 2$	'add 2'	20, 22, 24, . . .	
$Q_o = 20, Q_{n+1} = Q_n - 2$	'subtract 2'	20, 18, 16, . . .	

As a general rule, if D is a constant, a recurrence relation rule of the form:

$V_{n+1} = V_n + D$ can be used to model **linear growth**.

$V_{n-1} = V_n - D$ can be used to model **linear decay**.

In Chapters 6 and 7 we will use this knowledge to model and investigate simple interest loans and investments, as well as flat rate depreciation and unit cost depreciation of assets. But first, in the next section, we will look at graphing the first-order recurrence relations discussed above.